Alternative Orderings, Multicoloring Schemes, and Consistently Ordered Matrices

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Abstract

The use of multicoloring as a means for the efficient implementation of many diverse iterative methods for the solution of linear systems of equations, arising from the finite difference discretization of partial differential equations, on both parallel (concurrent) and vector computers has been extensive; these include both SOR-type and preconditioned conjugate gradient methods as well as smoothing procedures for use in multigrid methods. Multicolor orderings, corresponding to reorderings of the points of the discretization, often allow a local decoupling of the unknowns. Some new theory is presented which allows one to verify quickly whether or not a member of a certain class of matrices is consistently ordered (or π-consistently ordered) solely by looking at the structure of the matrix under consideration. This theory allows one to ascertain quickly that, while many well-known multicoloring schemes do give rise to coefficient matrices which are consistently ordered, many others do not. Examples are given from the literature, and other topics related to the theory are discussed. We survey many alternative orderings and multicoloring schemes proposed in the literature and apply the theory to the resulting coefficient matrices.

1 Introduction

The discretization by finite differences of elliptic partial differential equations often leads to the solution of linear systems of equations

$$Au = f. \tag{1}$$

With the advent of parallel, or concurrent, computers and vector processors, it has become apparent that the use of alternative orderings, i.e., other than the *natural* or *lexicographic ordering*, may increase efficiency in the implementation of many iterative methods for solving (1); these methods include, for example, the Jacobi, Gauss-Seidel, and SOR iterations and various preconditioned conjugate gradient methods, as well as smoothing procedures for use in multigrid methods. This leads naturally to the use of the technique of *multicoloring* in

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order to decouple the unknowns at the grid points of the discretization which was used to obtain the system (1). This is the subject of § 2.

Not too long ago a fair amount of attention was given to the concept of *consistently ordered (CO) matrices* (see § 3) and some generalizations thereof: generalized CO (GCO), CO(q, r) (see Appendix A), and π-CO matrices (Young [45]). Much of the foundation of the work done in this area can be found in the classical texts by Young [45] and Varga [39]. Lately, however, interest as to whether or not the coefficient matrix A of a particular system such as (1) is consistently ordered or not has somewhat waned. As a result one often works with a system of equations that is not CO (or GCO, π-CO, etc.) when a simple permutation of the elements of A might yield a matrix with one or more of these properties. This is, of course, not always a crucial consideration, but, at times, it can prove advantageous. The motivation for desiring that the coefficient matrix A have one or more of these properties is discussed in § 3 along with some concepts related to that of consistent ordering.

In § 4 and 5 we give some new theoretical results as to when matrices with a certain underlying block structure may be CO or π-CO ("block" CO). In § 6 we apply some of these results to show the consistent ordering of some standard alternative orderings and the lack of this property for some other orderings proposed in the literature. Finally, in § 7 we summarize our results. Appendix A contains some applications of the results to another class of matrices.

2 Multicoloring

Multicoloring is a technique that can allow for the local decoupling of the unknowns at the grid points of a finite difference, or finite element, discretization of a partial differential equation. The basic idea is to "color" the grid points in such a way that unknowns corresponding to grid points of a particular color are coupled only with unknowns of other colors. Thus, it is possible to update all unknowns of a single color simultaneously, i.e., in parallel, or with a single vector instruction, assuming the unknowns are stored appropriately.

In general, multicoloring corresponds to a partitioning π of the coefficient matrix into the block $p \times p$ form

$$A_p = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdot & \cdot & A_{1,p} \\ A_{2,1} & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & A_{p-1,p} \\ A_{p,1} & \cdot & \cdot & A_{p,p-1} & A_{p,p} \end{bmatrix}, \tag{2}$$

where the diagonal blocks A_{ii} are square. For a multicoloring scheme which gives a decoupling of the unknowns the A_{ii} are also diagonal. In most instances, a significant number of the off-diagonal blocks A_{ij} contain only zero entries.

In (2) p is the number of colors. Throughout this paper we shall maintain the notational convention that a *single* subscript on a matrix name is used to emphasize the *block* order of that matrix; when necessary for clarity, the corresponding partitioning π will carry the same subscript as in "π_p."

Multicoloring has been used ubiquitously for the solution of linear systems by iterative methods on both parallel and vector computers (for a review, see, e.g., Ortega and Voigt [29], also the bibliography Ortega *et al.* [30]). For instance, Lambiotte [21] used a red/black ordering to vectorize efficiently the SOR method. For the efficient implementation of the

SOR method on parallel computers Adams and Ortega [4] give a general treatment of multicoloring with $p > 2$ colors (see also Adams [1]). O'Leary [27] presents some other orderings which are amenable to the parallel implementation of a block SOR method.

The technique of multicoloring is not limited in application strictly to SOR-type iterative methods; it can also prove to be a valuable tool for use in conjunction with preconditioned conjugate gradient (PCG) methods. Poole and Ortega [36] use multicoloring to carry out the incomplete Cholesky conjugate gradient (ICCG) method on vector computers, and Poole [35] gives many examples of multicoloring schemes and the associated structure of the coefficient matrices corresponding to a mixed derivative problem and a plane stress problem; the theory to be presented in later sections allows one to ascertain, virtually at a glance, that none of the multicoloring schemes proposed there give rise to consistently ordered or even π_p-consistently ordered matrices ($p \geq 2$). Harrar and Ortega [19] solved a three-dimensional generalized Poisson equation using a red/black ordering to vectorize efficiently the SSOR iteration for use in a PCG method; Harrar [16] elucidates the considerable computational savings that can be obtained using this ordering with this particular method. The parallel and vector implementation of the SSOR PCG method (as well as the SOR method) using "many-color red/black" orderings (Harrar [16]) for both two- and three-dimensional problems is discussed in Harrar and Ortega [17]; these can be viewed as generalizations of an ordering proposed by Melhem [24] and will be discussed in more detail in § 6.2.

Multigrid methods (Brandt [6]) have proven to be useful and efficient for a wide variety of problems. When applying multigrid methods to the solution of elliptic problems, much of the computation time is spent on the relaxation procedure used at each grid level. Multicoloring is useful in this area as well. Gauss-Seidel smoothing with a red/black ordering has been shown to be quite effective (Foerster, *et al.* [13]). Alternating direction line methods are particularly robust; line methods correspond to a class of multicoloring strategies (see § 6.1). Also, *zebra* orderings (§ 6.1), can be particularly effective for anisotropic equations (Stuben and Trottenberg [38]). See Brandt [7] for further discussion on parallelizing multigrid relaxation procedures for use on concurrent computers.

3 Consistently ordered matrices and related concepts

In this section we review some background concepts such as consistently ordered matrices, compatible ordering vectors, π-consistently ordered matrices, and T-matrices. The definitions given are those of Young [45]. We also introduce the concept of a π-compatible ordering vector.

3.1 Motivation for consistent ordering

One property that may or may not obtain for the coefficient matrix A as a result of a reordering of the unknowns is that of being a *consistently ordered matrix.*

Definition 1 *The matrix A of order n is* consistently ordered (CO) *if for some t there exist disjoint subsets $S_1, S_2, ..., S_t$ of $W = \{1, 2, ..., n\}$ such that $\bigcup_{k=1}^{t} S_k = W$ and such that, if $a_{ij} \neq 0$, then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$, where S_k is the subset containing i.*

We remark that some, though not all, of the S_k may be empty.

Determination of the optimum relaxation parameter

$$\omega_{opt} = \frac{2}{1+(1-\mu_r{}^2)^{\frac{1}{2}}}, \quad \mu_r = \rho(J) \tag{3}$$

for the SOR method applied to the system (1) via Young's classical SOR theory (Young [45]) is based upon the relationship

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2 \tag{4}$$

between the eigenvalues μ of the Jacobi iteration matrix associated with A

$$J = D^{-1}(L+U) = I - D^{-1}A, \tag{5}$$

and the eigenvalues λ of the corresponding SOR iteration matrix

$$\mathcal{L}_\omega = (D-\omega L)^{-1}[(1-\omega)D + \omega U]. \tag{6}$$

In (5) and (6) above

$$A = D - L - U \tag{7}$$

is the splitting of A into its diagonal, strictly lower triangular, and strictly upper triangular parts, respectively. The derivation of (4) is, in turn, based upon the invariance of

$$\Delta = det(\alpha L + \alpha^{-1}U - kD) \tag{8}$$

with respect to α for all $\alpha \neq 0$ and for all k; this is true for the class of matrices known as T-matrices (Young [45], Theorem 2.1, pg. 141). (For the definition of a T-matrix, see Definition 5 in § 4.) CO matrices were introduced as a more general class of matrices for which the determinantal invariance (8), and hence (4) and (3), hold (in fact, the invariance of (8) is at times used to *define* CO matrices as in Ortega [28]).

CO matrices also possess ***property A*** as defined by Young [45]; in fact, for any matrix possessing property A, there exists a suitable permutation of the rows and columns of A such that one can obtain a CO matrix (Young [45], Theorem 4.5, pg. 150).

Varga [39] was able to extend Young's result to obtain

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p \tag{9}$$

relating the eigenvalues of a CO weakly cyclic Jacobi iteration matrix J (of index $p \geq 2$) to the eigenvalues of the corresponding SOR iteration matrix $\mathcal{L}_\omega$.

Young [44] conjectured that, under certain additional assumptions, orderings resulting in CO coefficient matrices were optimal in that they give the fastest rate of convergence for the SOR method with relaxation parameter $\omega = 1$, i.e., the Gauss-Seidel method. The main result of Varga [40] contains as a special case a proof of this conjecture. With $\mathcal{L}_\omega$ defined by (6), ω_{opt} given by (3), and letting ρ denote spectral radius, he showed that under certain conditions, $\rho(\mathcal{L}_{\omega_{opt}}) > \omega_{opt} - 1$ and $\min_{0<\omega<2} \rho(\mathcal{L}_\omega) > \omega_{opt} - 1$, unless A is CO.

The usefulness of this theory is not limited to the case that one is using the SOR method to solve (1). For example, Harrar and Ortega [18] used the fact that a matrix with (2-cyclic) red/black form (44) is CO and a result of Wachspress [43] relating the eigenvalues of $\mathcal{L}_\omega$ to those of $\mathcal{S}_\omega$, the SSOR (symmetric SOR) iteration matrix, to derive an optimality result for the relaxation parameter ω in the context of the m-step SSOR preconditioned

conjugate gradient method. They showed that $\omega = 1$ is optimal, in the sense of minimizing the condition number of the preconditioned system, for any number m of preconditioning steps. However, the effect of consistent ordering, if any, on the rate of convergence of SSOR PCG methods is not known. Duff and Meurant [11] investigated the effect of ordering on the ICCG method; they considered seventeen different orderings, some of which are easily analyzed with respect to consistent (and/or π-consistent) ordering using the theory to be presented in the sequel (see § 6.3).

As mentioned in § 1, there are several generalizations of consistent ordering. One of these is that of generalized (q,r)-consistent ordering (GCO(q,r)) (Young [45]). ((q,r)-consistently ordered matrices are discussed briefly in Appendix A.) Such matrices are p-cyclic, i.e., the Jacobi iteration matrix J associated with A is weakly cyclic of index p (Varga [39]). Recently analysis on convergence domains for the SSOR iteration for certain classes of GCO(q,r) matrices has been done by Varga, *et al.* [42] and Neumaier and Varga [25]. Chong and Cai [8] showed that for GCO$(k, p-k)$ matrices the functional equation

$$[\lambda_s - (\omega-1)^2]^p = \lambda_s^k[\lambda_s - (\omega-1)]^{p-2k}(2-\omega)^{2k}\omega^p\mu^p \tag{10}$$

holds between the Jacobi eigenvalues μ and the eigenvalues λ_s of the SSOR iteration matrix associated with A; the parameter k has to do with the locations of the nonzero blocks in the p-cyclic matrix A. Neumaier and Varga [42] derived (10) for the case $k = 1$ (we note that their derivation, unlike most for this type of result, does not make use of a determinantal invariance such as (8).) Hadjidimos and Neumann [14], [15] used this result to derive convergence and divergence domain results for the SSOR method applied to this class of matrices. Finally, we note that Li and Varga [22] very recently derived a new functional equation which generalizes and unifies *all* of the recent research articles on the SSOR and USSOR methods applied to p-cyclic matrices (USSOR is "unsymmetric SOR," in which one uses different relaxation parameters, say ω_f and ω_r, for the forward and reverse SOR sweeps, respectively). For the SSOR method their result reduces to (10).

3.2 Compatible ordering vectors, π-consistently ordered matrices, and T-matrices

Although it is possible to show consistent ordering directly by appealing to Definition 1, i.e., by finding the sets S_k, it is often more convenient to make use of the notion of a *compatible ordering vector.*

Definition 2 *The vector* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ *where the* γ_i *are integers is a* compatible ordering vector *for the matrix* A *of order* n *if, for* $a_{ij} \neq 0$,

$$\gamma_i - \gamma_j = \begin{cases} 1 & \text{if} \quad i > j, \\ -1 & \text{if} \quad i < j. \end{cases} \tag{11}$$

The usefulness of compatible ordering vectors is made clear by the following theorem:

Theorem 1 *A matrix A is consistently ordered if, and only if, a compatible ordering vector exists for A.*

(For a proof see Young [45], Theorem 3.2, pg. 146.)

In the sequel we will be concerned primarily with the property of being CO for block $p \times p$ matrices of the form (2). To this end we have a weaker version of consistent ordering given by

Definition 3 *Let the matrix A be partitioned as in (2) and define a $p \times p$ matrix $Z = (z_{rs})$ by*

$$z_{rs} = \begin{cases} 0 & \text{if} \quad A_{rs} = 0, \\ 1 & \text{if} \quad A_{rs} \neq 0. \end{cases} \tag{12}$$

*The matrix A is π_p-*consistently ordered (π_p-CO) *if Z is consistently ordered.*

We note that, according to our notation, an $n \times n$ CO matrix is also π_n-CO.

Analogous to Definition 2 for a compatible ordering vector we now introduce the (new) concept of a *π_p-compatible ordering vector* for a block $p \times p$ matrix.

Definition 4 *The vector $\gamma = (\gamma_1, \gamma_2, ..., \gamma_p)^T$ where the γ_i are integers is a π_p-*compatible ordering vector *for the block $p \times p$ matrix A if, for Z, corresponding to A, as defined in (12), (11) holds.*

We state without proof the following analog of Theorem 1.

Theorem 2 *A block $p \times p$ matrix A is π_p-consistently ordered if, and only if, there exists for A a π_p-compatible ordering vector.*

Obviously, a matrix of the form (2) can be π-CO and still not be CO. For example, a full 4×4 matrix partitioned as a block 2×2 matrix is π_2-CO but is not CO since we cannot construct a compatible ordering vector for it. This is because in such a matrix we have $a_{ij} \neq 0$ for all indices i, j, and it is easily seen that

Observation 1 *It is impossible to construct a (π-)compatible ordering vector for a matrix of (block) order greater than two all of whose elements are nonzero; that is, no (block) full matrix of (block) order greater than two is (π-)consistently ordered.*

We also note that

Observation 2 *All matrices of order greater than one are π_2-consistently ordered.*

One type of matrix of the form (2) which is both π-CO and CO is a *T-matrix.*

Definition 5 *A matrix with the block tri-diagonal form*

$$T_p = \begin{bmatrix} D_1 & H_1 & & & \\ K_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & H_{p-1} \\ & & & K_{p-1} & D_p \end{bmatrix}, \tag{13}$$

*where the D_i are square diagonal matrices is a T_p-*matrix.

With Theorems 3 and 4 we show that T_p-matrices are both π_p-CO and CO. That a T_p-matrix is π_p-CO is a corollary of

Theorem 3 *A block $p \times p$ tri-diagonal matrix is a π_p-consistently ordered matrix.*

proof: We assert that $\gamma = (1, 2, ..., p)^T$ is a π_p-compatible ordering vector for a matrix of the form (13) where the D_i need not be diagonal. In the matrix Z of (12) corresponding to a matrix of this form, the only off-diagonal nonzero elements are $z_{i,i+1}$ and $z_{i+1,i}$ where $i = 1, ..., p-1$. Thus, in the construction of a compatible ordering vector for Z we require

only that $\gamma_{i+1} - \gamma_i = 1$, $i = 1, \ldots, p-1$. Clearly, the elements of the vector γ satisfy this condition, and hence γ is a compatible ordering vector for Z and, in turn, a π_p-compatible ordering vector for a matrix of the form (13) where the D_i are not necessarily diagonal. Therefore, by Theorem 2, a block $p \times p$ tri-diagonal matrix is π_p-CO. □

Now, although a block tri-diagonal matrix is not, in general, CO, if its diagonal blocks are diagonal matrices, i.e., if the matrix is a T-matrix, then we have the following result.

Theorem 4 *A T-matrix is a consistently ordered matrix.*

(For a proof see Young [45], Theorem 3.1, pg. 145.)

The property of being π-CO is important because it is often the case that although a given matrix may not be CO, it is π-CO for some partitioning π of A into blocks. And, in such circumstances, one may apply the results of the classical SOR theory to the partitioned matrix, i.e., one obtains the eigenvalue relation (4) where μ and λ are now the eigenvalues of the Jacobi and SOR iteration matrices, respectively, corresponding to the block partitioning of A. That is, the Jacobi and SOR iteration matrices are still given by (5) and (6), respectively, except now (7) is the splitting of A into its block diagonal, block strictly lower triangular, and block strictly upper triangular parts. This allows, for example, the computation of the optimal relaxation parameter ω_{opt} and the corresponding spectral radius ρ_{opt} for line SOR methods (see § 6.1).

In subsequent sections we search for classes of more general block matrices (having more than just three nonzero block diagonals) which are also CO or at least π-CO.

4 The addition of block bi-diagonal matrices B_r to T_p

In the present section we seek to investigate what types of matrices can be added to block tri-diagonal matrices (and hence also to T-matrices) so that the resulting matrix sum is still π-CO. In this way, we will find other classes of matrices, which have an underlying block tri-diagonal structure, that are also π-CO.

This section is divided into three subsections. In the first we examine the case that the block tri-diagonal matrix has no zero blocks on either the first sub- or super-diagonal, that is, $H_i \neq 0$ or $K_i \neq 0$ for $i = 1, \ldots, p-1$ in (13). In the second subsection we consider the case that the block $p \times p$ tri-diagonal matrix has intermittent zero blocks on these diagonals, say $H_{kq} = 0$, $K_{kq} = 0$ for $k = 1, \ldots, r$ where $r = p/q$ and q is some integer which divides p evenly; the generalization to the case for which these zero blocks are spaced non-uniformly should be obvious. Finally, in the last subsection we consider the case that the block tri-diagonal matrix is a T-matrix. In this case we will find that we can say something about such a matrix being CO, not just π-CO.

Of course, it is important to keep in mind throughout this paper that all results on π-consistent ordering for block matrices inherently have concomitant corollaries for the case of consistent ordering of non-block matrices as a result of the correspondence implied in Definition 3. That is, all results for block $p \times p$ matrices with blocks A_{ij} in terms of π-consistent ordering imply exactly analogous results for $n \times n$ (where $n = p$) matrices with elements a_{ij} in terms of consistent ordering.

4.1 T_p has no zero blocks on the first sub- or super-diagonal

Here we consider a certain class of block $p \times p$ tri-diagonal matrices T_p; namely, those that have no zero blocks on either the first sub-diagonal or the first super-diagonal. We want to

know what type of block $p \times p$ matrices A_p, if any, can be added to T_p so that the resulting matrix sum $T_p + A_p$ is π_p-CO. We will find that A_p must, in fact, also be block tri-diagonal; this is shown to be the case in the following theorem.

Theorem 5 *Let T_p be a block tri-diagonal matrix of the form (13) where the D_i are not necessarily diagonal, and suppose A_p has the block $p \times p$ form (2) where the blocks are partitioned commensurately with those of T_p. If $H_i \neq 0$ and $H_i \neq -A_{i,i+1}$, or $K_i \neq 0$ and $K_i \neq -A_{i+1,i}$, for $i = 1, ..., p-1$, then the matrix $T_p + A_p$ is a π_p-CO matrix if, and only if, $A_{ij} = 0$ if $i > j+1$ or $j > i+1$, i.e., A_p is block tri-diagonal.*

proof: Clearly the sum of block tri-diagonal matrices is also block tri-diagonal so that if A_p is then so is $T_p + A_p$, and therefore $T_p + A_p$ is π_p-CO by Lemma 3.

Now, assume $T_p + A_p$ is π_p-CO and suppose $A_{ij} \neq 0$ for some i, j with, without loss of generality, $j > i+1$. Since $T_p + A_p$ is π_p-CO, we can, by Theorem 2 construct a π_p-compatible ordering vector γ for $T_p + A_p$. Now, either $[T_p + A_p]_{i,i+1} = H_i + A_{i,i+1} \neq 0$ or $[T_p + A_p]_{i+1,i} = K_i + A_{i+1,i} \neq 0$, for $i = 1, ..., p-1$ since $H_i \neq -A_{i,i+1}$ or $K_i \neq -A_{i+1,i}$, respectively. Therefore, we must have $\gamma_{i+1} - \gamma_i = 1$ for $i = 1, ..., p-1$. Now, $[T_p + A_p]_{ij} = A_{ij} \neq 0$ so that we also require $\gamma_j - \gamma_i = 1$. However,

$$\gamma_j - \gamma_i \;=\; (\gamma_j - \gamma_{i+1}) + (\gamma_{i+1} - \gamma_i) \;=\; (\gamma_j - \gamma_{i+1}) + 1 > 1$$

since $j > i+1$, a contradiction. Therefore, we must have $A_{ij} = 0$ for $j > i+1$. The case $A_{ij} \neq 0$ with $i > j+1$ is exactly analogous. □

We note that, under the assumptions of Theorem 5, if T_p and A_p are T-matrices then $T_p + A_p$ will be CO by Lemma 4 since the sum of T-matrices is a T-matrix. However, the corresponding "only if" part of the theorem will not hold, in general, for T-matrices and consistent ordering. Although it is possible to add matrices other than T-matrices to a T-matrix to obtain a CO matrix, the only type we can add *without knowing anything about the internal structure of the off-diagonal blocks of T and A* is a T-matrix.

Note that an immediate corollary of Theorem 5 is

Corollary 1 *Let the $n \times n$ matrix A be such that all of the elements on the first sub- or super-diagonal are nonzero. Then A is CO if, and only if, $a_{ij} = 0$ for $j > i+1$ and $i > j+1$.*

4.2 T_p has intermittent zero blocks on the first sub- and super-diagonal

We now consider block tri-diagonal matrices which have intermittent zero blocks on the first off-diagonals. That is, we let the matrix T have the block $r \times r$ diagonal form

$$T \;=\; diag(T_{11}, T_{22}, ..., T_{rr}), \tag{14}$$

where the $q \times q$ diagonal blocks T_{kk} are of the form

$$T_{kk} \;=\; \begin{bmatrix} A_{l+1,l+1} & A_{l+1,l+2} & & & \\ A_{l+2,l+1} & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & A_{l+q-1,l+q} \\ & & & A_{l+q,l+q-1} & A_{l+q,l+q} \end{bmatrix}, \tag{15}$$

for $k = 1, ..., r$, where $l = (k-1)q$ and $p = qr$. According to our notational convention, $T = T_r = T_p$, where T_r is block $r \times r$ uni-diagonal with $q \times q$ blocks and T_p is block $p \times p$

tri-diagonal with zero blocks every qth entry on the first sub- and super-diagonal. Since T_p is block tri-diagonal, it is π_p-CO by Lemma 3 (it is also trivially π_r-CO). Of course, if the diagonal blocks $A_{l+i,l+i}$, $i = 1, ..., q$ are square diagonal matrices, i.e., $A_{l+i,l+i} = D_{l+i}$, in (15) then T_p given by (14), (15) is also a T_p-matrix and hence CO by Theorem 4.

Now consider the class of block $r \times r$ bi-diagonal matrices of the form

$$B_r^m = \begin{bmatrix} 0 & B_{1,2}^m & & & \\ B_{2,1}^m & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & B_{r-1,r}^m \\ & & & B_{r,r-1}^m & 0 \end{bmatrix}, \tag{16}$$

where the $q \times q$ nonzero blocks $B_{k,k+1}^m$, $k = 1, ..., r-1$ either have only one nonzero block lower diagonal ($m = m_L$)

$$B_{k,k+1}^{m_L} = L_{k,k+1}^{m_L} = \begin{bmatrix} & & & \\ A_{l+m_L,kq+1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & A_{kq,(k+1)q-(m_L-1)} \end{bmatrix}, \tag{17}$$

where $m_L = 2, ..., q$, or a nonzero block main diagonal ($m = m_D$)

$$B_{k,k+1}^{m_D} = D_{k,k+1} \tag{18}$$

(note that $m = m_D$ is equivalent to the case $m_L = 1$ in (17)), or one nonzero block upper diagonal ($m = m_U$)

$$B_{k,k+1}^{m_U} = U_{k,k+1}^{m_U} = \begin{bmatrix} A_{l+1,kq+m_U} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ & & & & A_{kq-(m_U-1),(k+1)q} \\ & & & & \end{bmatrix}, \tag{19}$$

where $m_U = 2, ..., q$. The blocks $B_{k+1,k}^m$, $k = 1, ..., r-1$ have the same block structure as the blocks $(B_{k,k+1}^m)^T$. Note that T_p is block tri-diagonal of block order p while B_r^m is block bi-diagonal of block order $r = p/q$; the block orders are different.

We now have the following lemma.

Lemma 1 *Let T_p be given by (14), (15) and B_r^m be given by (16) with (17), (18), or (19). Then the matrix $T_p + B_r^m$ is π_p-CO for all values of*

$$m = \begin{cases} m_L = 2, ..., q, \\ m_D = 1, \\ m_U = 2, ..., q. \end{cases} \tag{20}$$

proof: We show that the matrices $T_p + B_r^m$ are π_p-CO by showing the existence of π_p-compatible ordering vectors γ. We treat separately the cases that the blocks $B_{k,k+1}^m$ have the form (17), (18), or (19), i.e., $m = m_L$, $m = m_D$, or $m = m_U$, respectively. First, assume that the $B_{k,k+1}^m = L_{k,k+1}^{m_L}$ so that they have the form (17). In this case one can verify that a π_p-compatible ordering vector for $T_p + B_r^{m_L}$ is given by

$$\begin{aligned}\gamma^L &= [1,...,q,\ m_L+1,...,m_L+q,\ 2m_L+1,...,2m_L+q,\ \cdots,\\ &\qquad (r-1)m_L+1,...,(r-1)m_L+q]^T\end{aligned}$$

or, in a somewhat more compact form notationally,

$$\gamma^L = [((k-1)m_L+1,(k-1)m_L+2,...,(k-1)m_L+q),k=1,...,r]^T. \tag{21}$$

If $B_{k,k+1}^m = D_{k,k+1}$, i.e., $m = m_D = 1$, we substitute 1 for m_L in (21) to obtain

$$\gamma^D = [(k,k+1,...,k+q-1),k=1,...,r]^T \tag{22}$$

as a π_p-compatible ordering vector for $T_p + B_r^{m_D}$. Now, suppose $m = m_U$, i.e., $B_{k,k+1}^m = U_{k,k+1}^{m_U}$ has the form (19). Then we obtain the π_p-compatible ordering vector

$$\gamma^U = [((k-1)(m_U-2)+1,(k-1)(m_U-2)+2,...,(k-1)(m_U-2)+q),k=1,...,r]^T \tag{23}$$

for $T_p + B_r^{m_U}$. We note that for the case $m_U = 2$, i.e., the $B_{k,k+1}^{m_U}$ are nonzero only on the first block superdiagonal, we obtain $\gamma^U = [(1,...,q),k=1,...,r]^T$ as a π_p-compatible ordering vector. One may verify that the vectors γ^L, γ^D, and γ^U given by (21), (22), and (23), respectively, are π_p-compatible ordering vectors for the matrices $T_p + B_r^m$ where the blocks $B_{k,k+1}^m$ of B_r^m have the form (17), (18), and (19), respectively. Thus, by Theorem 2, $T_p + B_r^m$ is π_p-CO for all values of m given by (20). □

The regularity among the elements of the π_p-compatible ordering vectors for the matrices $T_p + B_r^m$ is quite striking. It is important to note that the elements of these vectors corresponding to a given block $B_{k,k+1}^m$ are consecutive integers beginning with the $(k-1)q+1$st element of the vector; this is true for $k = 1,...,r$. That is, we have

$$\gamma_{(k-1)q+i} = \gamma_{(k-1)q+i-1}+1,\ \ i=2,...,q,\ \ k=1,...,r. \tag{24}$$

Therefore, for a given k, the only element of γ which depends on elements corresponding to another value of k is the $(k-1)q+1$st, the rest of the elements for that given k can be obtained using (24). This suggests that it may be possible to construct π_p-compatible ordering vectors for matrices $T_p + B_r$ (T_p given by (14), (15)) where the matrix B_r now has the somewhat more general block $r \times r$ bi-diagonal form

$$B_r = \begin{bmatrix} 0 & B_{1,2}^{m_1} & & & \\ B_{2,1}^{m_1} & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & B_{r-1,r}^{m_{r-1}} \\ & & & B_{r,r-1}^{m_{r-1}} & 0 \end{bmatrix}, \tag{25}$$

where each of the $q \times q$ nonzero blocks $B_{k,k+1}^{m_k}$, $i = 1,...,r-1$ has one of the three forms (17), (18), or (19); that is, each m_k can take on any value m in (20). We shall see in Lemma 3 that we can indeed construct π_p-compatible ordering vectors for these matrices $T_p + B_r$ thereby showing that they are π_p-CO by Theorem 2. However, in the proof of that lemma we need the following

Lemma 2 *If γ is a (π-)compatible ordering vector for a matrix A, then so is $\beta = \gamma + \delta$ where δ is any constant vector.*

proof: Note that for any elements β_i, β_j of $\beta = \gamma + \delta$ we have

$$\beta_i - \beta_j = (\gamma_i + \delta_i) - (\gamma_j + \delta_j) = (\gamma_i - \gamma_j) - (\delta_i - \delta_j) = \gamma_i - \gamma_j$$

since δ is a constant vector. Similarly $\beta_j - \beta_i = \gamma_j - \gamma_i$. Thus, since γ is a (π-)compatible ordering vector for A, (11) holds with γ_i, γ_j replaced by β_i, β_j. Thus, β is a (π-)compatible ordering vector for A. □

We can now prove

Lemma 3 *Let T_p be given by (14), (15) and let B_r be given by (25) with each $B^{m_k}_{k,k+1}$ given by one of (17), (18), or (19), and where each m_k takes on a value of m from (20). Then the matrix $T_p + B_r$ is π_p-CO for all choices of B_r, i.e., for all combinations of the m_k.*

proof: Using the π_p-compatible ordering vectors which we constructed in the proof of Lemma 1 we will show how to construct a π_p-compatible ordering vector for $T_p + B_r$, given any B_r of the form (25). Consider the block $B^{m_k}_{k,k+1}$. If this block has the form (17) so that $m_k = m_L$ for some m_L, then we notice from (21) that in going from γ^L_{kq} to γ^L_{kq+1} we need only add m_L to $\gamma^L_{(k-1)q+1}$. Similarly, if $B^{m_k}_{k,k+1}$ has the block diagonal form (18), we obtain γ^D_{kq+1} from $\gamma^D_{(k-1)q+1}$ by adding $m_D = 1$ to $\gamma^D_{(k-1)q+1}$. If $B^{m_k}_{k,k+1}$ has the block strictly upper triangular form (19), we see that γ^U_{kq+1} can be obtained by subtracting $m_U - 2$ from $\gamma^U_{(k-1)q+1}$. In summary then, we have

$$\gamma_{kq+1} = \begin{cases} \gamma_{(k-1)q+1} + m_k, & \text{if } m_k = m_L, \\ \gamma_{(k-1)q+1} + 1, & \text{if } m_k = m_D, \\ \gamma_{(k-1)q+1} - (m_k - 2), & \text{if } m_k = m_U. \end{cases} \tag{26}$$

Of course, as noted above, the remaining elements are contiguous and would be calculated using (24). Now, the construction of a π_p-compatible ordering vector for $T_p + B_r$ proceeds as follows. First, we take as the first q elements of the vector γ the integers $1, ..., q$, i.e., $\gamma_i = i, i = 1, ..., q$. Next we consider the block $B^{m_1}_{1,2}$. If this block is block strictly lower triangular, then we set $\gamma_{q+1} = \gamma_1 + m_1 = 1 + m_1$, $\gamma_{q+i} = \gamma_{q+i-1} + 1, i = 2, ..., q$. Similarly, if $B^{m_1}_{1,2}$ is of the form (18), i.e., $m_1 = 1$, we use (26) with $k = 1$ to obtain γ_{q+1} and calculate the next $q - 1$ elements using (24). However, if $B^{m_1}_{1,2}$ is block strictly upper triangular we cannot simply use (26) to calculate γ_{q+1} since for $m_1 > 2$ we would obtain a value for γ_{q+1} which was non-positive. In order to mitigate this problem, we would add $m_1 - 2$ to the first q elements of γ then calculate γ_{q+1} using (26) and the next $q - 1$ elements again using (24); we can do this since we know from Lemma 2 that if γ is a π-compatible ordering vector then so is $\gamma + \delta$ where δ is any constant vector. With the blocks $B^{m_k}_{k,k+1}$, $k = 2, ..., r-1$ we proceed in exactly the same fashion obtaining γ_{kq+1} from $\gamma_{(k-1)q+1}$ using equations (26) with $m = m_k$ and then using (24) to calculate the next $q - 1$ elements of γ. Of course, if $B^{m_k}_{k,k+1}$ has the form (19), we first check to be sure that $\gamma_{kq+1} = \gamma_{(k-1)q+1} - (m_k - 2) > 0$; if not then we first add $m_k - 2$ to the thus far computed kq elements of γ. When we have proceeded through all of the blocks $B^{m_k}_{k,k+1}$, we have constructed a π_p-compatible ordering vector for $T_p + B_r$, thus, by Theorem 2, $T_p + B_r$ is π_p-CO. □

We again emphasize that it is important to remember that all of the results of this section can be used to show consistent ordering for matrices of the form $T_p + B_r$ (or $T_p + B^m_r$) if the $A_{l+i,l+j}$ of (15) and the $A_{l+i,kq+j}$ of (17)-(19) are 1×1 matrices, i.e., scalar elements.

We now show that, if B_r is of the form (25), then $T_p + B_r$ cannot be π_p-CO unless each of the blocks $B^{m_k}_{k,k+1}$, $k = 1, ..., r-1$, has one of the uni-diagonal forms (17)-(19).

Theorem 6 *Let T_p be given by (14), (15) and let B_r be given by (25). Then the matrix sum $T_p + B_r$ is π_p-CO if, and only if, each $B^{m_k}_{k,k+1}$, $k = 1, ..., r-1$ is given by one of (17), (18), (19), where each m_k takes on a value of m from (20).*

proof: If each $B^{m_k}_{k,k+1}$ is given by one of (17), (18), or (19) with m_k taking on a value of m from (20), then $T_p + B_r$ is π_p-CO by Lemma 3.

We now show that if any of the blocks $B^{m_k}_{k,k+1}$ has a form different from one of the uni-diagonal forms (17)-(19), then it is impossible to construct a π_p-compatible ordering vector for $T_p + B_r$; thus, the matrix sum $T_p + B_r$ is not π_p-CO by Theorem 2. Consider the block $B^{m_k}_{k,k+1}$ where k is now fixed and is chosen from the range of values $k = 1, ..., r-1$. We assert that given one nonzero block $A_{l+i,kq+j}$ (recall that $l = (k-1)q$), where $i, j \in \{1, ..., q\}$, in $B^{m_k}_{k,k+1}$, the only other blocks of $B^{m_k}_{k,k+1}$ which can be nonzero are those lying along the diagonal of which $A_{l+i,kq+j}$ is a member. We shall treat two cases: Case 1. $(i \geq j)$: $A_{l+i,kq+j}$ is in the lower triangular portion of $B^{m_k}_{k,k+1}$ or on the main diagonal of $B^{m_k}_{k,k+1}$ and Case 2. $(i < j)$: $A_{l+i,kq+j}$ is in the upper triangular portion of $B^{m_k}_{k,k+1}$.

Case 1. $(i \geq j)$: From (17) we see that $B^{m_k}_{k,k+1} = B^{m_L}_{k,k+1}$ where $m_L = i - j + 1$ and $A_{l+i,kq+j} \neq 0$ for some i, j where $m_L \leq i \leq q$ and $1 \leq j \leq q - (m_L - 1)$. We claim that, in order that a π_p-compatible ordering vector exist for $T_p + B_r$, the only other allowable nonzero blocks in $B^{m_k}_{k,k+1}$ will lie on the diagonal of which the block $A_{l+i,kq+j}$ is a member. The parameter m_L uniquely determines this diagonal, and it can be seen that all elements of this diagonal are of the form $A_{l+i,kq+j}$ where the integer pair (i, j) is a member of the set

$$\Lambda_{m_L} = \{ (i,j) | m_L \leq i \leq q, 1 \leq j \leq q - (m_L - 1) \, and \, i - j = m_L - 1 \} . \tag{27}$$

Assume now that, for some $\xi, \eta \in \{1, ..., q\}$, $A_{l+\xi,kq+\eta} \neq 0$ and (ξ, η) is not an element of Λ_{m_L}. Then in order that a π_p-compatible ordering vector exist, the requirement (11) becomes

$$\gamma_{l+\xi} - \gamma_{kq+\eta} = \begin{cases} 1 & \text{if} \quad l + \xi > kq + \eta, \\ -1 & \text{if} \quad l + \xi < kq + \eta. \end{cases}$$

However, since $\xi, \eta \in \{1, ..., q\}$, we can never have $l + \xi > kq + \eta$. Noting that $kq - l = kq - (k-1)q = q$ so that $l + \xi < kq + \eta$ is always satisfied, we thus require

$$\gamma_{l+\xi} - \gamma_{kq+\eta} = -1. \tag{28}$$

Now, since the elements of γ are consecutive for indices from $l + 1$ to kq,

$$\begin{aligned} \gamma_{l+\xi} &= (\gamma_{l+\xi} - \gamma_{l+\xi-1}) + (\gamma_{l+\xi-1} - \gamma_{l+\xi-2}) + ... + (\gamma_{l+2} - \gamma_{l+1}) + \gamma_{l+1} \\ &= (\xi - 1) + \gamma_{l+1}. \end{aligned} \tag{29}$$

Similarly, the elements of γ are consecutive for indices from $kq + 1$ to $(k+1)q$ so that

$$\begin{aligned} \gamma_{kq+\eta} &= (\gamma_{kq+\eta} - \gamma_{kq+\eta-1}) + (\gamma_{kq+\eta-1} - \gamma_{kq+\eta-2}) + ... + (\gamma_{kq+2} - \gamma_{kq+1}) + \gamma_{kq+1} \\ &= (\eta - 1) + \gamma_{kq+1}. \end{aligned} \tag{30}$$

Subtracting (30) from (29) and using the first line of (26), we have

$$\gamma_{l+\xi} - \gamma_{kq+\eta} = (\xi - \eta) + \gamma_{l+1} - \gamma_{kq+1} = (\xi - \eta) - m_L. \tag{31}$$

Substituting into (28) this gives

$$\xi - \eta \;=\; m_L - 1.$$

But then from the definition of Λ_{m_L} we would have $(\xi, \eta) \in \Lambda_{m_L}$, a contradiction. Therefore we must have $A_{l+\xi,kq+\eta} = 0$ for (ξ, η) not in Λ_{m_L}.

Case 2. $(i < j)$: In this case we assume that the block $B^{m_k}_{k,k+1}$ contains a nonzero block $A_{l+i,kq+j}$ where now $1 \leq i \leq q - (m_U - 1)$ and $m_U \leq j \leq q$. Thus, $B^{m_k}_{k,k+1}$ has the form $B^{m_U}_{k,k+1}$ where $m_U = j - i + 1$. Then, in order that a π_p-compatible ordering vector exist for $T_p + B_r$, we assert that all other nonzero blocks of this $B^{m_k}_{k,k+1}$ lie on the diagonal determined by m_U. From (19) we see that for any element $A_{l+i,kq+j}$ of this diagonal the integer pair (i, j) is a member of the set

$$\Upsilon_{m_U} \;=\; \{\, (i,j) | 1 \leq i \leq q - (m_U - 1), m_U \leq j \leq q \; and \; j - i = m_U - 1 \,\} \;. \qquad (32)$$

Now, assume that $A_{l+\xi,kq+\eta} \neq 0$ for some $\xi, \eta \in \{1, ..., q\}$ such that (ξ, η) is not in Υ_{m_U}. Then, analogous to Case 1, in order that a π_p-compatible ordering vector exist for $T_p + B_r$, we again obtain the requirement (28). Using (29) and (30) and the third line of (26), from which $\gamma_{l+1} - \gamma_{kq+1} = m_U - 2$, (31) becomes

$$\gamma_{l+\xi} - \gamma_{kq+\eta} \;=\; (\xi - \eta) + m_U - 2.$$

Substituting this into (28) we get

$$\xi - \eta \;=\; 1 - m_U$$

so that (ξ, η) is in Υ_{m_U}, a contradiction. Thus, $A_{l+\xi,kq+\eta} = 0$ for (ξ, η) not in Υ_{m_U}. Therefore, we conclude that, in order that a π_p-compatible ordering vector exist for $T_p + B_r$ so that $T_p + B_r$ is π_p-CO, the nonzero blocks $B^{m_k}_{k,k+1}$ of B_r must, for each k, have one of the uni-diagonal forms (17), (18), or (19), and the proof is complete. □

4.3 T_p is a T_p-matrix

Now, we consider the case that T_p is a T_p-matrix with intermittent zeroes on the first sub- and super-diagonal. Of course, in this case a matrix of the form $T_p + B_r$ will still be π_p-CO by Theorem 6, but it turns out that such a matrix is also CO.

Theorem 7 *Suppose the diagonal blocks $A_{l+i,l+i}$ of (15) are such that*

$$A_{l+i,l+i} \;=\; D_{l+i}\,, \;\; i = 1, ..., q, \qquad (33)$$

where the D_{l+i} are square diagonal matrices, and let B_r be a block bi-diagonal matrix of the form (25). Then any matrix of the form $T_p + B_r$ where T_p is given by (14), (15), and (33) is CO if each of the blocks $B^{m}_{k,k+1}$ of B_r is given by one of the uni-diagonal forms (17), (18), or (19).

proof: If each $B^{m_k}_{k,k+1}$ is given by one of (17), (18), or (19) with m_k taking on a value of m from (20), then we will show that we can easily construct a compatible ordering vector for $T_p + B_r$ using our previous results. Let s denote the order of the diagonal blocks given by (33). (The case that the diagonal blocks D_{l+i} each have different order, say s_{l+i}, is no more difficult to prove; however, the subscripting may be confusing enough as it is, thus, for ease of notation, we assume the order of these blocks is constant.) We assert that in

order to construct a compatible ordering vector γ for $T_p + B_r$ all we need do is take the π_p-compatible ordering vector (which we now denote γ^π) for $T_p + B_r$, constructed as in the proof of Lemma 3, and repeat each element s times. That is, with $l = (k-1)q$ we set

$$\gamma_{(l+i-1)s+\hat{\imath}} = \gamma^\pi_{l+i}, \quad \hat{\imath} = 1, ..., s; \quad i = 1, ..., q; \quad k = 1, ..., r. \tag{34}$$

In order to verify that γ constructed in this manner is a compatible ordering vector for $T_p + B_r$, we must show that (11) holds for all nonzero elements a_{gh} of $T_p + B_r$; note that $g, h \in \{1, ..., n\}$ where $n = qrs$. There are two situations that we must treat: $a_{gh} \neq 0$ represents an element of one of the off-diagonal blocks $A_{l+i,l+i+1}$ of (15) and $a_{gh} \neq 0$ represents an element of some $A_{l+i,kq+j}$ of $B^{m_k}_{k,k+1}$. Consider the case that $a_{gh} \neq 0$ is an element of one of the off-diagonal blocks of (15), $A_{l+i,l+i+1}$. Then we have

$$g = (l+i-1)s + \hat{\imath}, \quad h = (l+i)s + \hat{\jmath},$$

where $\hat{\imath}, \hat{\jmath} \in \{1, ..., s\}$. So, the requirement (11) becomes

$$\gamma_g - \gamma_h = \gamma_{(l+i-1)s+\hat{\imath}} - \gamma_{(l+i)s+\hat{\jmath}} = \begin{cases} 1 & \textit{if} \quad (l+i-1)s + \hat{\imath} > (l+i)s + \hat{\jmath}, \\ -1 & \textit{if} \quad (l+i-1)s + \hat{\imath} < (l+i)s + \hat{\jmath}. \end{cases}$$

Since $\hat{\imath}, \hat{\jmath} \in \{1, ..., s\}$, we can never have $(l+i-1)s + \hat{\imath} > (l+i)s + \hat{\jmath}$ so we require

$$\gamma_{(l+i-1)s+\hat{\imath}} - \gamma_{(l+i)s+\hat{\jmath}} = -1.$$

By (34) this is equivalent to requiring that

$$\gamma^\pi_{l+i} - \gamma^\pi_{l+i+1} = -1,$$

which is true in our construction of γ^π by (24).

Now, we consider the case that $a_{gh} \neq 0$ is in a block $A_{l+i,kq+j}$ of $B^{m_k}_{k,k+1}$. Then we have

$$g = (l+i-1)s + \hat{\imath}, \quad h = (kq+j-1)s + \hat{\jmath}, \tag{35}$$

for some $\hat{\imath}, \hat{\jmath} \in \{1, ..., s\}$. Proceeding as before, we obtain the requirement

$$\gamma_{(l+i-1)s+\hat{\imath}} - \gamma_{(kq+j-1)s+\hat{\jmath}} = -1.$$

By (34) we thus require

$$\gamma^\pi_{l+i} - \gamma^\pi_{kq+j} = -1. \tag{36}$$

This is the requirement (28) with $\xi = i$, $\eta = j$ which we found to hold if and only if $(\xi, \eta) = (i, j)$ is in Λ_{m_L} or Υ_{m_U}, depending on whether $m_k = m_L$ (or m_D) or $m_k = m_U$; that is, if and only if $A_{l+i,kq+j}$ lies along the diagonal determined by m_k. This is true by assumption, thus (11) holds for γ_g and γ_h corresponding to a_{gh}.

So, for any nonzero element of either of the two "types" of nonzero blocks ($A_{l+i,l+i+1}$ and $A_{l+i,kq+j}$) of $T_p + B_r$, the corresponding elements of γ given by (34) satisfy the requirements set forth in Definition 2 of a compatible ordering vector. Therefore, using (34), where the elements γ^π_{kq+i} are found as in Lemma 3, we can construct a compatible ordering vector for $T_p + B_r$ so it is a CO matrix by Theorem 1. □

As was the case with Theorem 5 in § 4.1, we cannot strengthen the above result to be an "if and only if" statement. It is, however, possible to show that an ordering vector *constructed as in the proof of Theorem 7* will be a compatible ordering vector for $T_p + B_r$

if and only if each $B^{m_k}_{k,k+1}$ of B_r has one of the block uni-diagonal forms (17)-(19). To prove this one would combine elements of the proofs of Theorems 5 and 7 as follows: As in the proof of Theorem 5 we would treat the cases $i \geq j$ and $i < j$ separately. For the case $i \geq j$ we would do exactly as in the proof of Theorem 5 up to the sentence after the definition of Λ_{m_L} given by (27). However, now we note that the assumption given in that sentence would mean that $a_{gh} \neq 0$ for some g, h given by (35) with i, j replaced by ξ, η and $\hat{\imath}, \hat{\jmath} \in \{1, ..., s\}$. Using exactly the same logic that follows equation (35) we are lead to the requirement (36), again with i, j replaced by ξ, η. We notice that in the notation of the proof of Theorem 7 this is exactly equation (28) of the proof of Theorem 5. The remainder of the case $i \geq j$ would then proceed exactly as in the proof of Theorem 5 from equation (28) onward, resulting in the desired contradiction. The proof of the case $i < j$ would be done in an analogous manner.

In summary then we have that, although the blocks $B^{m_k}_{k,k+1}$ of B_r do not necessarily have to have one of the block uni-diagonal forms (17)-(19) in order that the matrix sum $T_p + B_r$ (with T_p a T_p-matrix) be CO, we can only use the method of constructing a compatible ordering vector given in the proof of Theorem 7 if the $B^{m_k}_{k,k+1}$ do have one of the uni-diagonal forms. Otherwise, we need to know something about the internal structure of the $A_{l+i,l+i+1}$ of (15) and the $A_{l+i,kq+j}$ of $B^{m_k}_{k,k+1}$.

We note that the method of proving Theorem 7 can be extended in an obvious way to prove the stronger and very useful result:

Theorem 8 *Let the block $p \times p$ matrix A be given by (2), and suppose the diagonal blocks A_{ii}, $i = 1, ..., p$ are diagonal matrices. If A is π_p-CO, then A is CO.*

A proof of Theorem 8 would go something like the following. If A is π_p-CO, then by Theorem 2 there exists for A a π_p-compatible ordering vector, say γ^{π_p}. Let s_i denote the order of the diagonal block A_{ii}, for $i = 1, ..., p$. Now construct a vector γ by repeating each element $\gamma^{\pi_p}_i$ of γ^{π_p} s_i times. Then, using the method of proof used for Theorem 7, we would show that the vector γ, consisting of $\sum_{i=1}^{p} s_i$ elements, is a compatible ordering vector for A. Thus, A is CO by Theorem 1.

Often, when using a multicoloring scheme, one obtains a block matrix in which the diagonal blocks are diagonal matrices. When this is the case, Theorem 8 can provide an efficient way of showing whether or not that matrix is CO by simply finding a π_p-compatible ordering vector for it rather than a compatible ordering vector. In general, this should represent a substantial simplification.

We note that the following Corollary of Theorem 8 may also be useful.

Corollary 2 *Let the block $p \times p$ matrix A have diagonal blocks A_{ii}, $i = 1, ..., p$ which are block diagonal of block order s. Assume that A can also be partitioned as a block $t \times t$ matrix with $t = ps$. If A is π_p-CO, then A is π_t-CO.*

We use this result in § 6.1 to show that zebra orderings result in π-CO matrices.

5 The addition of more general block matrices M_r to T_p

In the present section we consider the addition of a more general class of block matrices M_r to block tri-diagonal matrices T_p where the M_r have more than just two nonzero block

diagonals. We consider matrices M_r of the form

$$M_r = \begin{bmatrix} 0 & M_{1,2} & \cdot & \cdot & M_{1,r} \\ M_{2,1} & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & M_{r-1,r} \\ M_{r,1} & \cdot & \cdot & M_{r,r-1} & 0 \end{bmatrix}, \tag{37}$$

where we again assume that this matrix is symmetrically structured. That is, the nonzero block structure of a block M_{ij} is the same as that of M_{ji}^T. In the language of previous sections these matrices would be referred to as block "$(2r-2)$-diagonal" matrices. (We note that the assumption of symmetric structure is not a restrictive one; such a structure is obtained for all discretizations based on *SO-stencils* as defined by Adams and Jordan [2].)

As was the case with the block bi-diagonal matrices B_r (and B_r^m) of § 4 we have again assumed all diagonal blocks contain only zero entries. Of course, this is not essential; all results we obtain are also valid if the diagonal blocks of M_r are nonzero. If T_p is assumed to be a T_p-matrix, again all of the results hold if the diagonal blocks of M_r are nonzero, so long as they are diagonal. However, since the focus of this research is not specifically what types of block matrices we can add to T_p in order that the resulting matrix be (π_p-)CO, but rather what structure beyond that of T_p is allowed so that this property obtains, we assume for simplicity that these diagonal blocks of M_r contain only zero entries.

In § 4.1 we found that given a block tri-diagonal matrix T_p with no zero blocks on either the first sub- or super-diagonal, all blocks outside of the main diagonal and these first off-diagonals had to be zero if the matrix were to be π_p-CO. However, in § 4.2 we found that in the case that there were intermittent zero blocks on these two diagonals we could add to this matrix any matrix from a certain class of block bi-diagonal matrices and still maintain the property of being π_p-CO. It turns out that the π_p-compatible ordering vectors constructed in the manner prescribed there allow for even more nonzero blocks in the matrix sum; these nonzero blocks must again take one of the uni-diagonal forms (17)-(19).

We denote the off-diagonal blocks of the matrix M_r of (37) by $M_{k,k+i}$, where $k = 1, ..., r$ and $i = 1, ..., r-k$; associated with each of these blocks will be a value $m_{k,k+i}$, selected from equation (20), which will determine the uni-diagonal structure. Thus, i serves as an index for the super-diagonal (and, by the symmetrical structure assumption, the associated sub-diagonal) under consideration; the case $i = 1$ was the subject of § 4.

We begin with the case $i = 2$. There are several possible situations. We will see that the allowable value of $m_{k,k+2}$ depends on the values of $m_{k,k+1}$ and $m_{k+1,k+2}$; that is, $m_{k,k+2}$ depends on the values of m in the block to the left and in the block below the $k, k+2$nd block. There are four possibilities depending on whether these values of m are of the form m_L or m_U (throughout this section we shall subsume the case $m = m_D = 1$ into the m_L's, i.e., $m_D = 1 \Rightarrow m_L = 1$). The four possibilities, and the corresponding allowable values of $m_{k,k+2}$ in order that we maintain π_p-consistent ordering, are given below.

i. $m_{k,k+1} = m_L = 1, ..., q$ and $m_{k+1,k+2} = m_L = 1, ..., q$

$$\begin{aligned} m_{k,k+1} + m_{k+1,k+2} \le q &\implies m_{k,k+2} = m_L = m_{k,k+1} + m_{k+1,k+2}, \\ m_{k,k+1} + m_{k+1,k+2} > q &\implies M_{k,k+2}^{m_{k,k+2}} = 0. \end{aligned} \tag{38}$$

ii. $m_{k,k+1} = m_U = 2, ..., q$ and $m_{k+1,k+2} = m_U = 2, ..., q$

$$\begin{aligned} m_{k,k+1} + m_{k+1,k+2} - 2 \le q &\implies m_{k,k+2} = m_U = m_{k,k+1} + m_{k+1,k+2} - 2, \\ m_{k,k+1} + m_{k+1,k+2} - 2 > q &\implies M_{k,k+2}^{m_{k,k+2}} = 0. \end{aligned} \tag{39}$$

iii. $m_{k,k+1} = m_L = 1, \ldots, q$ and $m_{k+1,k+2} = m_U = 2, \ldots, q$

$$\begin{aligned} m_{k,k+1} > m_{k+1,k+2} - 2 &\implies m_{k,k+2} = m_L = m_{k,k+1} - m_{k+1,k+2} + 2, \\ m_{k,k+1} \leq m_{k+1,k+2} - 2 &\implies m_{k,k+2} = m_U = m_{k+1,k+2} - m_{k,k+1}. \end{aligned} \tag{40}$$

iv. $m_{k,k+1} = m_U = 2, \ldots, q$ and $m_{k+1,k+2} = m_L = 1, \ldots, q$

$$\begin{aligned} m_{k+1,k+2} \leq m_{k,k+1} - 2 &\implies m_{k,k+2} = m_U = m_{k,k+1} - m_{k+1,k+2}, \\ m_{k+1,k+2} > m_{k,k+1} - 2 &\implies m_{k,k+2} = m_L = m_{k+1,k+2} - m_{k,k+1} + 2. \end{aligned} \tag{41}$$

Now, for values of i greater than 2, it turns out we need to use the values of any pair of m's, $m_{k,k+t}$ and $m_{k+t,k+i}$, where $0 < t < i$. For instance, to obtain the allowable value of, say $m_{k,k+4}$, we could use any of the pairs $m_{k,k+1}, m_{k+1,k+4}$ or $m_{k,k+2}, m_{k+2,k+4}$ or $m_{k,k+3}, m_{k+3,k+4}$. The formulas to obtain the allowable value of $m_{k,k+i}$ are exactly those given above with $k+1$ replaced by $k+t$ and $k+2$ replaced by $k+i$. Note, however, that when a block was set to zero as in the second lines of (i) and (ii) above, we do not assume a value of 0 for m in that block, rather m would carry the value indicated in the corresponding first line, although greater than q.

In this manner, it is possible to check a block matrix for the property of being π_p-CO. If all of the diagonal blocks $A_{l+i,l+i}$ are diagonal matrices then, by Theorem 8, the matrix can be checked for the property of being CO.

6 Application of the theory to some multicolor orderings

In this section we apply some of the results of § 4 to some well-known and not so well-known orderings which appear in the literature. In order to concretize some of what follows, we shall apply the discussion to the solution of the two-dimensional Laplace problem

$$\begin{aligned} \nabla^2 u &= 0 \quad in \quad \Omega = [0,1] \times [0,1], \\ u &= 0 \quad on \quad \partial\Omega. \end{aligned} \tag{42}$$

We assume that there are an even number N of grid points in each of the two coordinate directions in the discretization of (42).

First we will consider the so-called *line orderings* and also *zebra orderings*. Next we discuss the row-wise and plane-wise red/black orderings of Harrar and Ortega [17]. Harrar [16] gives this group of orderings the somewhat oxymoronic name of "many-color red/black orderings" for reasons that should become clear from the presentation given in § 6.2. Finally, in the last subsection we consider some other orderings proposed in the literature.

6.1 Line and zebra orderings

One frequently used class of orderings is the class of *line orderings*. These are multicoloring schemes in which all of the points on a given line of the grid, or group of lines, has the same color. Thus, for example, for a two-dimensional problem on an $N \times N$ grid, a 1-line ordering results in N colors, while a k-line ordering gives N/k colors (one, in general, chooses k so that it divides N evenly).

It is easy to show that such multicoloring schemes, corresponding to different partitionings π, give rise to π-CO matrices for five-point finite difference discretizations of (42) (and seven-point discretizations of its three-dimensional analog). If we order the lines, or groups

of lines, in the natural ordering from bottom to top, then the coefficient matrix will, as with a natural ordering of the grid points, have the form

$$A = \begin{bmatrix} T & -I & & & \\ -I & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -I \\ & & & -I & T \end{bmatrix}, \quad T = \begin{bmatrix} 4 & -1 & & & \\ -1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 4 \end{bmatrix}. \tag{43}$$

Here the matrix T is $N \times N$, and I is the identity matrix of order N. Different values of k, corresponding to different numbers of lines of a single color, will give rise to different partitionings of this matrix. For $k = 1$, we have a block $N \times N$ structure. For general k, A would be partitioned as a block $\frac{N}{k} \times \frac{N}{k}$ matrix of $kN \times kN$ blocks. In each case, the matrix is block tri-diagonal of block order N/k and hence is $\pi_{N/k}$-CO by Theorem 3. (We note that in the literature, when discussing line orderings, the subscript on π usually gives the number of lines per group, e.g., π_k is the partitioning according to groups of k lines and π_0 denotes the natural ordering. However, here we maintain the convention used throughout this paper that the subscript denotes the block order of the corresponding coefficient matrix.)

It is well known that the use of block iterative methods often results in faster convergence rates. For the problem (42) the line Jacobi and line Gauss-Seidel methods will converge twice as fast as the corresponding point methods; similarly, line SOR with optimal relaxation parameter will converge approximately $2^{1/2}$ times as quickly as the point SOR method (Young [45]). For more on the rate of convergence of k-line iterative methods see, e.g., Parter [31], [32] and Parter and Steuerwalt [34], [33].

As mentioned at the end of § 3, one important aspect of π-consistent ordering is that while A may not be CO, it may be π-CO for some partitioning π. This is often useful when the grid stencil for a particular problem contains more than five points (in the case of a two-dimensional problem). For example, while a nine-point grid stencil will not yield a CO matrix, it will, with a grouping by k lines, yield a $\pi_{N/k}$-CO matrix.

Now, consider a *zebra ordering.* We color all of the odd-numbered rows of the grid, say, black and all of the even-numbered rows white. Within each color we then number the grid points in the natural ordering. Using a five-point stencil in the discretization of (42), the coefficient matrix would have the red/black (block 2×2) form

$$A = \begin{bmatrix} D_1 & C \\ C^T & D_2 \end{bmatrix}, \tag{44}$$

where

$$D_1 = D_2 = \begin{bmatrix} T & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \quad T \end{bmatrix}, \quad C = \begin{bmatrix} -I & & & \\ -I & \cdot & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & -I & -I \end{bmatrix}, \tag{45}$$

and T is given in (43). The D_i and C are block $\frac{N}{2} \times \frac{N}{2}$, and T, I are $N \times N$.

Since A is π_2-CO with D_1, D_2 block diagonal, we can appeal to Corollary 2 and determine that the coefficient matrix A for a zebra ordering is also π_N-CO.

We note that k-line orderings will, in general, require the solution of more complicated systems of equations at each iteration since the diagonal blocks will not be diagonal. However, if we further impose a red/black ordering on each group of k lines (rows) of the

grid, then these diagonal blocks will take on the red/black form (44) where D_1 and D_2 are $\frac{kN}{2} \times \frac{kN}{2}$ and diagonal. Such orderings are the subject of the next subsection.

6.2 Many-color red/black orderings

Although the rate of convergence of some iterative methods such as SOR may be enhanced by the use of multicoloring, this is not true for others such as the SSOR preconditioned conjugate gradient (PCG) method. For example, for the red/black ordering, the optimum relaxation parameter, in the sense of reducing the condition number of the preconditioned system, is $\omega_{opt} = 1$ for any number m of preconditioning steps as shown by Harrar and Ortega [18]. Solving a three-dimensional generalized Poisson equation using red/black SSOR with $\omega = 1$ as a preconditioner for the conjugate gradient method, Harrar and Ortega [19] found the rate of convergence to be considerably worse than that of natural order SSOR with experimentally determined $\omega_{opt} > 1$. These results are consistent with similar experiments by Adams [1], Ashcraft and Grimes [5], Duff and Meurant [11], Elman and Agron [12], Melhem [24], and Poole and Ortega [36] for preconditioning by SSOR or ILU factorization.

Harrar and Ortega [17] (see also Harrar [16]) proposed a compromise between the faster convergence rate obtained with the natural ordering and the superior degree of parallelism and/or vectorization provided by the red/black ordering. Their *many-color red/black orderings* are of two general types: row-wise red/black orderings and planar red/black orderings. The *1-row* and *2k-row red/black orderings*, involve imposing a red/black ordering on every row or $2k$ rows (lines) of the grid where $k = 1, 2, ..., N/2$ and N is the number of rows. For three-dimensional problems one can also consider *1-plane* and *2k-plane red/black orderings* where a red/black ordering is imposed on every plane or $2k$ planes of the grid, respectively. In either case, the red and black unknowns of a given color are numbered in a natural ordering, and the vector of unknowns is then given by

$$u = (u_{R1}, u_{B1}, u_{R2}, u_{B2}, u_{R3}, u_{B3}, ...)^T. \tag{46}$$

For example, a 2-row red/black ordering of a 6×6 grid is shown in Figure 1.

B3	R3	B3	R3	B3	R3
R3	B3	R3	B3	R3	B3
B2	R2	B2	R2	B2	R2
R2	B2	R2	B2	R2	B2
B1	R1	B1	R1	B1	R1
R1	B1	R1	B1	R1	B1

Figure 1: 2-row red/black ordering of a 6×6 grid

This ordering can be viewed as having been obtained by first introducing a 2-line ordering, then imposing a red/black ordering on each group of 2 lines.

In general, for a $2k$-row red/black ordering the linear system will be of the form

$$\begin{bmatrix} D_1 & A_{12} & 0 & A_{14} & \cdot & \cdot & \cdot \\ A_{12}^T & D_2 & A_{23} & 0 & \cdot & \cdot & \cdot \\ 0 & A_{23}^T & D_3 & A_{34} & 0 & A_{36} & \\ A_{14}^T & 0 & A_{34}^T & D_4 & A_{45} & 0 & \\ \cdot & \cdot & 0 & A_{45}^T & \cdot & & \\ \cdot & \cdot & A_{36}^T & & & \cdot & \\ \cdot & \cdot & & & & & \cdot \end{bmatrix} \begin{bmatrix} u_{R1} \\ u_{B1} \\ u_{R2} \\ u_{B2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = f. \tag{47}$$

Here the D_i are diagonal matrices and u_{Ri} is the vector of unknowns associated with the Ri grid points, and similarly for u_{Bi}. For an $N \times N$ grid the coefficient matrix will be block $\frac{N}{k} \times \frac{N}{k}$, where each block is $kN \times kN$. The internal structure of the off-diagonal blocks is given in Harrar and Ortega [17] and Harrar [16] and is not pertinent to this research. For three-dimensional problems on an $N \times N \times N$ grid, a $2k$-plane red/black ordering will again yield a system of the form (47), where the coefficient matrix is block $\frac{N}{k} \times \frac{N}{k}$, except now the blocks will be $kN^2 \times kN^2$ and consist of more nonzero diagonals.

The coefficient matrix of the system (47) is a matrix of the form $T_p + A_p$ where $p = N/k$; T_p is a T_p-matrix (a $T_{N/k}$-matrix), and A_p is a block $\frac{N}{k} \times \frac{N}{k}$ matrix which has all zero blocks except for $A_{i,i+3}$ (and $A_{i+3,i} = A_{i,i+3}^T$) where $i = 1, 3, ..., N-3$. All of the blocks on the first sub- and super-diagonal of T_p are nonzero. Thus, by Theorem 5, $T_p + A_p$ is not π_p-CO. In order to determine whether or not the coefficient matrix of (47) is CO we would need to investigate the internal structure of its off-diagonal blocks.

Consider now a 1-row red/black ordering obtained by imposing a red/black ordering on each row of the grid. In this case the system of equations corresponding to a five-point finite difference discretization of (42) would have the block $2N \times 2N$ form

$$\begin{bmatrix} D_1 & A_{12} & A_{13} & 0 & \cdot & \cdot & \cdot \\ A_{12}^T & D_2 & 0 & A_{24} & \cdot & \cdot & \cdot \\ A_{13}^T & 0 & D_3 & A_{34} & A_{35} & & \\ 0 & A_{24}^T & A_{34}^T & D_4 & 0 & & \\ \cdot & \cdot & A_{35}^T & & \cdot & & \\ \cdot & \cdot & & & & \cdot & \\ \cdot & \cdot & & & & & \cdot \end{bmatrix} \begin{bmatrix} u_{R1} \\ u_{B1} \\ u_{R2} \\ u_{B2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = f, \tag{48}$$

where each block is $\frac{N}{2} \times \frac{N}{2}$. This is also the form of the system of equations for the three-dimensional analog of (42) in the case of a 1-plane red/black ordering except that each block would be $\frac{N^2}{2} \times \frac{N^2}{2}$ and the $A_{i,i+1}$, $i = 1, ..., N-1$ would have more nonzero diagonals.

Returning to the nomenclature of § 4, the 1-row (or 1-plane) red/black coefficient matrix of (48) is a matrix of the form $T_p + B_r$, where $p = 2N$, $r = N$ (for two-dimensional problems and $p = 2N^2$, $r = N^2$ for three dimensional problems). T_p is block tri-diagonal with intermittent zero blocks every second block on the first sub- and super-diagonal. B_r is block bi-diagonal where each block $B_{k,k+1}^m$ is block 2×2 with $m = m_D = 1$. This matrix is thus π_p-CO by Lemma 3. In fact, from the proof of that lemma we see that a π_p-compatible ordering vector for this matrix is given by (22) with $q = 2$, that is, $\gamma^D = (1, 2, 2, 3, ..., 2N, 2N+1)^T$. (For the three-dimensional problem with a 1-plane red/black ordering we would replace N by N^2.) Further, since the diagonal blocks of T_p are diagonal matrices, we can use Theorem 7 to obtain as a compatible ordering vector for $T_p + B_r$

$$\gamma^D = [(1)_{N/2}, (2)_{N/2}, (2)_{N/2}, (3)_{N/2}, ..., (2N)_{N/2}, (2N+1)_{N/2}]^T,$$

where $(i)_s$ denotes the s-long vector all of whose elements are i. Thus, we know that the coefficient matrix of the system (48) is CO *without knowing anything about the internal structure of the off-diagonal blocks.*

Returning to the $2k$-row red/black orderings, suppose we simply reorder the unknowns in the vector u of (46) so that

$$u = (u_{R1}, u_{B1}, u_{B2}, u_{R2}, u_{R3}, u_{B3}, ...)^T. \tag{49}$$

Then, in the coefficient matrix of the system (47), the blocks $A_{i,i+3}$, $i = 1, 3, ..., N-3$, and their transposes, will move in one block closer to the main diagonal while the blocks $A_{i,i+1}$, $i = 2, 4, ..., N-2$, and their transposes, will move out one block. That is, the coefficient matrix would now have the same block form as the 1-row red/black matrix of (48). Thus, we again obtain a π_p-CO matrix which is, in fact, CO as well. We note that Harrar and Ortega [17] discussed this reordering as a way to reduce the bandwidth of the coefficient matrix, but made no mention of consistently (or π-consistently) ordered matrices.

Harrar and Ortega [17] used these orderings and the 1-step SSOR PCG method to solve a three-dimensional generalized Poisson equation on a Cray-2 (see Harrar [16] for extended results on a Cray-2 and also a CDC Cyber 205). They found a significant jump in the number of iterations until convergence when going from the 1-row to 2-row red/black ordering, and a slow increase in iterations when going from the 2-row to the 4-row to the 8-row, etc. red/black orderings. Then they found another jump when going to a 1-plane red/black ordering. Corresponding to the behavior of the iteration count was an inverse effect on the optimum value of the relaxation parameter ω. They had no explanation for this behavior, but perhaps it is related to the observation made here that none of the $2k$-row red/black orderings, as formulated there, yielded (π_p-)CO coefficient matrices.

6.3 Other orderings

In this section we discuss some other orderings which have been proposed in the literature. The primary motivation for many of these orderings is the need for more than two colors to decouple the unknowns under a nine-point grid stencil (for a two-dimensional problem); in this case it is well known that at least four colors are necessary. Adams and Jordan [2] identified seventy-two distinct four-color orderings which could be used to bring about this local decoupling of unknowns. They grouped these orderings into equivalence classes for which the convergence behavior is the same. For a nine-point discretization of the model problem (42) their theory gives six such classes. Adams *et al.* [3] analyzed four of these classes and found they could be reduced to two (they were unable to analyze the other two classes with the proposed methods). All seventy-two of these four-color orderings lead to matrices which are neither CO nor π_4-CO.

Adams and Jordan [2] define a *multicolor*, or *c-color*, matrix to be a block matrix of the form (2) with $p = c$ and where the diagonal blocks A_{ii} are diagonal matrices. They also define a *multicolor T-matrix* as a block tri-diagonal matrix of the form

$$T_M = \begin{bmatrix} M_1 & L_1 & & & \\ U_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & L_{s-1} \\ & & & U_{s-1} & M_s \end{bmatrix}, \tag{50}$$

where the M_i are multicolor matrices, the L_i are block strictly lower triangular, and the U_i have the transposed structure of L_i, respectively. Now, by Theorem 8, if a multicolor matrix as defined above is π_c-CO, it is CO. However, by Theorem 5, if $A_{i,i+1} \neq 0$, $i = 1, ..., c-1$, then a multicolor matrix is not π_c-CO if any of the other A_{ij} are nonzero. If some of the $A_{i,i+1}$ do consist solely of zero entries, then Theorem 6 and the theory of § 5 may allow one to determine whether or not the multicolor matrix is π_c-CO. However, a multicolor T-matrix given by (50) is always π_s-CO by Theorem 3, and the main result of Adams and Jordan [2] is that a multicolor T-matrix and its associated multicolor matrix have corresponding SOR

iteration matrices with the same eigenvalues. Thus, we can apply the classical SOR theory to the π_s-CO multicolor T-matrix to gain information about the eigenvalues of the multicolor matrix which may be neither π-CO nor CO.

Kuo and Levy [20] also consider four-color ordering for a nine-point discretization of a two-dimensional Poisson equation. Rather than analyzing the Jacobi iteration matrix in the space domain, they consider a simpler, yet equivalent, four-color iteration matrix in the frequency domain. This matrix is not π_4-CO; however, they point out that it is π_2-CO (recall Observation 2). Hence, they apply the SOR theory to this frequency domain matrix partitioned as a block 2×2 matrix. They thus obtain what they call a "two-level four-color" SOR method in which they carry out a block (outer) SOR iteration at the first level followed by a point (inner) SOR iteration at the second level. The two-color (red/black) SOR method for a five-point grid stencil is shown to be a degenerate case of their general two-level four-color SOR method.

O'Leary [27] considered certain ordering schemes which would allow for the efficient implementation on parallel computers of iterative processes such as Gauss-Seidel and SOR. These schemes include the "P^3," "T^3," "$H+H$," "*Cross*," and "*Box*" orderings; the names are indicative of the patterns made by the blocks of grid points of a single color, and each ordering uses three colors. These ordering schemes give rise to coefficient matrices of the form (2) where each of the diagonal blocks A_{ii} is not a diagonal matrix. For example, partitioned as a block 3×3 matrix, the coefficient matrix corresponding to a P^3 ordering of the grid points (under a nine-point stencil) has diagonal blocks A_{ii} which are block diagonal. She gives the sparsity structure for this matrix, and it is immediately apparent that it is not π_3-CO by Theorem 5 since blocks $A_{1,2}$ and $A_{2,3}$ contain nonzero entries but so does block $A_{1,3}$. The grid corresponding to the sparsity structure pictured in that paper has five points "per P" if the block of points P is internal to the grid. Thus, many of the diagonal blocks internal to the A_{ii}, $i = 1,2,3$, are 5×5 and full. Therefore, this matrix is also not CO (see Observation 1). In order to analyze the rate of convergence of the SOR method in this ordering, she circumvents the difficulty of having a non-CO (and non-π-CO) matrix by using the "approximate" validity of the SOR theory for irreducible Stieltjes matrices; That is, $\omega_{opt} - 1 < \rho(\mathcal{L}_\omega) < \sqrt{\omega_{opt} - 1}$, where ω_{opt} is given by (3) (Young [45]).

Shortley and Weller [37] considered the use of $k \times k$ square blocks of points with the Gauss-Seidel method for the solution of (42). They found that $k \times k$ block Gauss-Seidel converged more quickly than point Gauss-Seidel by a factor of k; also $k \times k$ block Gauss-Seidel was found to be twice as fast as the corresponding Jacobi method. Parter and Steuerwalt [34] use "plane" iterative methods to solve a three-dimensional elliptic problem. They point out that the $k \times k$ block ordering, as well as the k-line ordering, lead to coefficient matrices which satisfy block property A (Young [45] refers to this as property $A^{(\pi)}$). In fact, these orderings give rise to π-CO matrices. We discussed this for the k-line orderings in § 6.1. Following the notational conventions of § 4, for a $k \times k$ block ordering one obtains a coefficient matrix of the form $T_p + B_r$ where T_p is a block $(\frac{N}{k})^2 \times (\frac{N}{k})^2$ tri-diagonal matrix of the form (14), (15) with zero blocks every N/k blocks on the first sub- and super-diagonal. B_r is block $\frac{N}{k} \times \frac{N}{k}$ bi-diagonal of the form (25) with nonzero blocks $B^{m_k}_{k,k+1}$ which are block $\frac{N}{k} \times \frac{N}{k}$ uni-diagonal with $m_k = m_D = 1$. Thus, the coefficient matrix corresponding to a $k \times k$ block ordering on an $N \times N$ grid is $\pi_{(\frac{N}{k})^2}$-CO by Theorem 6.

The effect of ordering on the rate of convergence of preconditioned conjugate gradient (PCG) methods is not well understood. Duff and Meurant [11] considered preconditioning by incomplete factorization (specifically, the ICCG(0) method of Meijerink and Van der

Vorst [23]) in seventeen different orderings, including the natural, red/black, zebra and four-color orderings already discussed. We now consider some of the other orderings.

Several diagonal orderings of the grid points, attributed to Cuthill and Mckee [9], are considered; these include forward and reverse diagonal orderings as well as a diagonal ordering of $k \times k$ blocks. On an $N \times N$ grid the forward and reverse diagonal orderings yield T_{2N-1}-matrices where the partitioning π_{2N-1} results in diagonal blocks of different orders, corresponding to the differing lengths of the $2N-1$ diagonals of the grid. Hence, these orderings yield π_{2N-1}-CO and CO matrices by Theorems 3 and 4, respectively. The block diagonal ordering gives rise to a block $(2\frac{N}{k}-1) \times (2\frac{N}{k}-1)$ tri-diagonal matrix where the diagonal blocks, again of varying orders, are block diagonal. Thus, the coefficient matrix is $\pi_{2\frac{N}{k}-1}$-CO by Theorem 3 and $\pi_{\frac{N^2}{k^2}}$-CO by the analog of Corollary 2 for the case that the block order of the ith diagonal block is $s = s(i) = \frac{N}{k} - |\frac{N}{k} - i| =$ the number of $k \times k$ blocks on the ith block diagonal of the grid. They also consider an "alternating diagonal ordering" which can be viewed as a zebra ordering of the grid by diagonals. This yields a π_2-CO coefficient matrix whose diagonal blocks are diagonal so it is CO by Theorem 8.

The "spiral ordering" (Duff, *et al.* [10]) results in "east-west" connections for all but the first and last grid points so that the coefficient matrix has an underlying tri-diagonal structure with no zeroes on either of the first two off-diagonals. Since there are other nonzero elements (from "north-south" connections), the matrix is therefore not CO by Corollary 1.

Two orderings attributed to Van der Vorst are obtained by first partitioning the grid into four $\frac{N}{2} \times \frac{N}{2}$ blocks. Then, starting with the point of each block which is a corner of the original grid, one obtains the first ordering by numbering the grid points of each block row-wise and the second ordering by taking the grid points of each block in a diagonal ordering. The two orderings are pictured in Figure 2.

1	2	3	21	20	19		1	3	6	24	21	19
4	5	6	24	23	22		2	5	8	26	23	20
7	8	9	27	26	25		4	7	9	27	25	22
16	17	18	36	35	34		13	16	18	36	34	31
13	14	15	33	32	31		11	14	17	35	32	29
10	11	12	30	29	28		10	12	15	33	30	28

Figure 2: a) Van der Vorst 1 and b) Van der Vorst 2 on a 6×6 grid

First we consider Van der Vorst 1 of Figure 2a. The coefficient matrix is of the form $T_4 + B_2$ where T_4 is block 4×4 tri-diagonal with zero blocks every $q = 2$ positions on the first sub- and super-diagonal. B_2 is block 2×2 with $B^{m_k}_{k,k+1} = B^{m_D}_{1,2}$ $(m_D = 1)$ so that $T_4 + B_2$ is π_4-CO by Theorem 6. Now, the coefficient matrix can also be partitioned as a matrix of the form $T_{2N} + M_4$ where T_{2N} is block tri-diagonal with zero blocks every $q = \frac{N}{2}$ positions on the first off-diagonals. In M_4 the blocks $M^{m_{12}}_{1,2}$ and $M^{m_{34}}_{3,4}$ are block uni-diagonal with $m_{12} = m_{34} = m_L = 1$, and $M_{2,3} = 0$; these are the allowable values given in (38). Finally, $M_{1,4} = 0$. Thus, the coefficient matrix $T_{2N} + M_4$ is π_{2N}-CO by the theory of § 5.

For Van der Vorst 2 (Figure 2b) the coefficient matrix is again π_4-CO using the same analysis as above. The matrix is also of the form $T_{4(N-1)} + M_4$ where $T_{4(N-1)}$ is block tri-diagonal with zero blocks every $q = N-1$ blocks on the first off-diagonals, and M_4 is such that its off-diagonal blocks $M^{m_{ij}}_{i,j}$ again satisfy the conditions set forth in § 5 showing that this ordering yields a matrix which is $\pi_{4(N-1)}$-CO.

7 Summary

Multicoloring provides a valuable technique to increase the efficiency of implementing certain iterative processes to solve linear systems of equations on both parallel and vector computers. These iterative processes include not only SOR-type methods, but also PCG methods as well as relaxation processes for use in multigrid methods.

Application of Young's classical SOR theory is valid when the coefficient matrix of the system to be solved is CO. Often, especially when a multicoloring scheme is introduced, one obtains a coefficient matrix which is not CO; however, this matrix may be π-CO (block CO) for some partitioning π.

The determination of whether or not a given matrix is CO or π-CO is often nontrivial. We have presented some theory which allows one to ascertain quickly whether matrices which have an underlying block tri-diagonal structure are (π-)CO or not; such matrices are often obtained when a multicoloring scheme is used. We found that the only block $p \times p$ matrices with all nonzero blocks on the first sub- or super-diagonal which are π_p-CO are block tri-diagonal matrices. Of course, an obvious corollary is that tri-diagonal matrices are the only CO matrices which have no zeroes on either of the first off-diagonals.

We then found that if there were zero blocks on the first sub- and super-diagonal of A_p, we could have further off-diagonal nonzero blocks and still have a π_p-CO matrix. If the zero blocks on these first off-diagonals occurred every qth block, then we can have nonzero blocks in the $q \times q$ block $B^{m_k}_{k,k+1}$ whose $(q,1)$-block $A_{q,1}$ is the zero block on the first super-diagonal. However, we found that, in order for A_p to be π_p-CO, given one nonzero block $A_{l+i,kq+j}$ in $B^{m_k}_{k,k+1}$, all other nonzero blocks had to lie along the same block diagonal of which $A_{l+i,kq+j}$ is a member. In § 5 we showed that more nonzero block $q \times q$ uni-diagonal matrices could appear farther off the main diagonal.

Theorem 8 provides a way of verifying whether or not a block $p \times p$ matrix A_p whose diagonal blocks are diagonal is CO; we need only determine whether or not A_p is π_p-CO since we can use a π_p-compatible ordering vector to obtain a compatible ordering vector.

The theory was then applied to many examples of ordering schemes from the literature to show that while some commonly used orderings give rise to CO or π_p-CO ($p > 2$) matrices, many others do not. This was particulary true for multicolor orderings with more than two colors although the theory presented here can be used to show that the class of multicolor orderings introduced by Harrar and Ortega [17] will yield matrices which are not only π_p-CO with $p > 2$, but CO as well.

A $T_p(q,r)$-matrices and (q,r)-consistently ordering

As mentioned in § 1 there are several generalizations of the class of CO matrices other than that of the class of π-CO matrices. These include (q,r)-*consitently ordered*, *generalized* (q,r)-*consistently ordered*, and *generalized* π-*consistently ordered* matrices (Young [45]). In this appendix we will not treat either of these "generalized" versions, although we will try to give a few examples of the ways in which the results of § 4 and 5 can be used to obtain some results concerning (q,r)-consistently ordered ((q,r)-CO) matrices. In particular, these results will apply to matrices from the class of $T_p(q,r)$-*matrices*; this class represents a generalization of the class of T-matrices originally defined in § 3.2. We note that in this appendix q and r have no relation to the q and r of previous sections. When we mean q and r as used previously, we shall denote them by $\hat{q}$ and $\hat{r}$.

A formal definition of a (q,r)-CO matrix can be found in Young [45]. We note only that a $(1,1)$-CO matrix is a CO matrix in the sense of Definition 1 and that for (q,r)-CO matrices we have the following analog of the Jacobi-SOR eigenvalue relation (4) (Young [45])

$$(\lambda + \omega - 1)^s = \lambda^r \omega^s \mu^s, \quad s = q + r. \tag{51}$$

Analogous to this generalization of CO matrices, we generalize the concept of a T_p-matrix to obtain (Young [45])

Definition 6 *Let q and r be positive integers less than p. The matrix A is a $T_p(q,r)$-matrix if it can be partitioned into the block $p \times p$ form $A = (A_{ij})$ where, for each i, $A_{ii} = D_i$ is a square diagonal matrix and where all other blocks vanish except possibly for the blocks $A_{i,i+r}$, $i = 1, 2, ..., p-r$ and $A_{i,i-q}$, $i = q+1, q+2, ..., p$.*

Clearly a $T_p(1,1)$-matrix is a T_p-matrix as given by Definition 5.

Just as T-matrices are CO, so too are $T(q,r)$-matrices (q,r)-CO. However, we will try to show under what circumstances $T_p(q,r)$-matrices are also π_p-CO and, since their diagonal blocks are diagonal matrices, CO by Theorem 8.

Now, note that, for the purposes of showing (π-)consistent ordering, a $T_p(q,r)$-matrix with either $q = 1$ or $r = 1$ can be treated as a matrix of the form $T_p + B_{\hat{r}}$ where T_p is a T_p-matrix and $B_{\hat{r}}$ is a block $\hat{r} \times \hat{r}$ bi-diagonal matrix with nonzero elements on its qth or rth sub- or super-diagonal, respectively. The case $q = 1$ is of particular importance since Varga [41] gave a complete analysis of this case and then Nichols and Fox [26] showed that the SOR method is not effective if $q > 1$. Also, the important class of p-cyclic matrices, $p \geq 2$ consists of matrices with nonzero diagonal elements which have a corresponding Jacobi iteration matrix that is permutationally similar to a $T_p(q,r)$-matrix where $r = p-1$ (thus the result (9) which is simply (51) with $r = p-1$, $s = q + r = 1 + (p-1) = p$). Therefore, in the sequel we consider only the case that one of q and r is unity.

Appealing to Theorem 5, we obtain our first result.

Theorem 9 *Let q and r be positive integers less than p. Suppose A is a $T_p(1,r)$-matrix such that $A_{i,i+r} \neq 0$, $i = 1, ..., p-r$ and $A_{i,i-1} \neq 0$, $i = 2, ..., p$. Similarly, suppose $\hat{A}$ is a $T_p(q,1)$-matrix such that $\hat{A}_{i,i+1} \neq 0$, $i = 1, ..., p-1$ and $\hat{A}_{i,i-q} \neq 0$, $i = q+1, q+2, ..., p$. Then A is π_p-consistently ordered if and only if $r = 1$, and $\hat{A}$ is π_p-consistently ordered if and only if $q = 1$.*

proof: Let A $(\hat{A})$ be as given in the hypothesis of the Theorem. Then A $(\hat{A})$ has no zero blocks on its first sub-(super-)diagonal. Hence, by Theorem 5, A $(\hat{A})$ can have no zero blocks outside the first off-diagonals. That is, we must have $r = 1$ $(q = 1)$.

Now, assume $r = 1$ $(q = 1)$. Then A $(\hat{A})$ is a $T_p(1,1)$-matrix, i.e. a T_p-matrix. Thus, by Theorem 3, A $(\hat{A})$ is π_p-CO. □

Since $T_p(q,r)$-matrices, by definition, have diagonal blocks which are diagonal matrices we have, by Theorem 8,

Corollary 3 *Assume the hypotheses of Theorem 9 hold. If $r = 1$ then A is consistently ordered. If $q = 1$ then $\hat{A}$ is consistently ordered.*

Of course, we are forced to drop the "only if" part of the result since we are making no assumptions about the internal structure of the off-diagonal blocks of A or $\hat{A}$.

Analogous to our progression in § 4, we now consider the case that the first sub-diagonal $(q = 1)$ or the first super-diagonal $(r = 1)$ has some zero blocks. The remainder of the results

of this appendix are stated only for the case $q = 1$, but it is trivial to adjust the proofs to handle the case $r = 1$; this should be clear from the proof of Theorem 9 given above.

First we treat the case that A is $T_p(1,r)$-matrix, with $r = p-1$, and has at least one zero block on the $q = 1$ sub-diagonal; in this case A is trivially a p-cyclic matrix.

Theorem 10 *Let A be a $T_p(1,p-1)$-matrix. A is π_p-consistently ordered if and only if $A_{i,i-1} = 0$ for some $i = 2,...,p$.*

proof: Assume $A_{k,k-1} = 0$ where $k = 2,...,p$ is fixed. The nonzero blocks of A are $A_{1,p}$ and $A_{i,i-1}$, $i = 2,...,k-1,k+1,...,p$. Thus, in the construction of a π_p-compatible ordering vector for A, we require $\gamma_p - \gamma_1 = 1$ and $\gamma_i - \gamma_{i-1} = 1$, where $i = 2,...,k-1,k+1,...,p$. One may easily verify that the elements of the vector

$$\gamma^{(\pi_p)} = (p, p+1, ..., p+k+1, k+2, k+3, ..., p+1)^T \tag{52}$$

satisfy both of these requirements. Thus, we have constructed a π_p-compatible ordering vector for A and A is π_p-CO by Theorem 2.

Now, let A be a π_p-CO $T_p(1,p-1)$-matrix and assume $A_{i,i-1} \neq 0$ for $i = 2,...,p$. By Theorem 9, a $T_p(1,r)$-matrix all of whose blocks on the first sub-diagonal are nonzero can be π_p-CO if and only if $r = 1$. However, we have $r = p-1$, a contradiction. Thus, we must have that one of $A_{i,i-1}$, $i = 2,...,p$ is zero. □

As with the previous Theorem, since the diagonal blocks of $T_p(q,r)$-matrices are diagonal matrices, we again obtain an immediate corollary.

Corollary 4 *Let A be a $T_p(1,p-1)$-matrix. If $A_{i,i-1} = 0$ for some $i = 2,...,p$ then A is consistently ordered.*

The above Theorem and Corollary are easily adapted to show that a $T_p(p-1,1)$-matrix A is (π_p-)CO if (and only if) $A_{i,i+1} = 0$ for some $i = 2,...,p$.

In § 4.2 we considered the case that T_p was tri-diagonal with intermittent zeroes every $\hat{q}$th entry on the first sub- and super-diagonal. We conclude this appendix with a natural extension of the results of that section. Recall that there we eventually allowed m_k, which determined the position of the sole nonzero block diagonal in the block $B^{m_k}_{k,k+1}$ of $B_{\hat{r}}$, to vary with each block. However, here we must keep m_k constant for all k so that the nonzero blocks $A_{l+i,kq+j}$ of all of the $B_{k,k+1}$ will lie along the same diagonal. This leads us to the final result of this appendix which we state without proof.

Theorem 11 *Let A be a $T_p(1,r)$-matrix with zero blocks every $\hat{q}$th position on the first ($q = 1$) sub-diagonal. Suppose $r = \hat{q} + m_U$ for some $m_U = 2,...,\hat{q}$ or $r = \hat{q} - (m_L - 2)$ for some $m_L = 1,...,\hat{q}$. If the rth super-diagonal has $m-1$ zero blocks following every $\hat{q} - (m-1)$ possibly nonzero blocks (where $m = m_U$ or $m = m_L$, depending on whether $r = r(m_U)$ or $r = r(m_L)$, respectively), then A is π_p-consistently ordered and consistently ordered.*

Of course, an analogous result holds for $T_p(q,1)$-matrices.

The results given here show that under certain conditions $T_p(q,r)$-matrices with either $q = 1$ or $r = 1$ are not only (q,r)-CO, but also π_p-CO and even CO.

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