An Upper Bound for the Linearized Map of an Inverse Problem for the Wave Equation

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AN UPPER BOUND FOR THE LINEARIZED MAP OF AN INVERSE PROBLEM FOR THE WAVE EQUATION [1]

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1 Introduction

A simplified model which governs many physical processes such as seismic and acoustic wave propagation is the following linear acoustic wave equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - \nabla\sigma\cdot\nabla\right)u = f\ , \tag{1.1}$$

where $\sigma = \sigma(x)$ is the logarithm of the density, $c = c(x)$ is the sound speed of the medium, and $f = f(x,t)$ is the source term which introduces the energy to the problem. If σ, c and f are given along with appropriate side conditions, the forward (or direct) problem is to determine $u = u(x,t)$, the excess pressure. For appropriate choices of σ, c, and f, u is determined uniquely by standard linear hyperbolic theory of partial differential equations (*p.d.e.*). Thus the problem stated above defines a map from the coefficients to the solution of the wave equation. In this paper, we study an aspect of the *regularity* of this map, or rather its composition with the trace on a time-like hypersurface.

Throughout this work we shall restrict ourselves to the special case of constant velocity c, though we believe that the ideas in this work may be extended to cover some more general cases.

To fix the ideas, write $x \in \mathbb{R}^n$ as (x', x_n), where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. We assume that the problem is set in the whole space $\mathbb{R}^n$ and $u = 0$ in the past $(t < 0)$. Take $f(x,t) = \delta(x,t)$ as an ideal point source. This assumption seems reasonable when the spatial extent of the source is much smaller than a typical wavelength and all frequency components to be measured are present in f. More explanations on the validity of these assumptions may be found in Symes [25]. Thus u is actually the retarded fundamental solution. We then have the following simple model:

$$\begin{aligned}\Box u \ &- \ \nabla\sigma\cdot\nabla u = \delta(x,t)\ ,\ \ (x,t)\in\mathbb{R}^n\times\mathbb{R} &(1.2)\\ u \ &= \ 0\ ,\ \ t<0\ , &(1.3)\end{aligned}$$

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where $\Box$ is defined to be $\partial_t^2 - \Delta$, and Δ is the Laplacian.

Define the *forward map* F as:

$$F : \sigma \longrightarrow (\phi u) \mid_{x_n=0} , \tag{1.4}$$

where $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$ and near$\{x_n = 0\}$. The reason for introducing this cut-off function, $\phi(x,t)$, is that we want to make sure the restriction of distribution u to the hypersurface $\{x_n = 0\}$ is well defined even though the equation (1.1) has a singular right-hand side.

Because F is nonlinear, one wants to work with the formal linearization (or formal derivative) DF, with respect to the reference state (σ_0, u_0), defined by first order perturbation theory (Born-approximation). We then have the following linearized problem

$$\begin{aligned} \Box\delta u \quad &- \quad \nabla\sigma_0 \cdot \nabla\delta u = \nabla\delta\sigma \cdot \nabla u_0 && (1.5) \\ \delta u \quad &= \quad 0\ ,\ \ t < 0\ . && (1.6) \end{aligned}$$

The formal derivative $DF(\sigma_0)$ is given by

$$DF(\sigma_0)\delta\sigma = (\phi\delta u) \mid_{x_n=0} . \tag{1.7}$$

It is our main goal in this work to determine the appropriate spaces of the domain and range of F for which

the formal derivative DF is bounded.

We believe that similar analysis will lead to the continuity or even differentiability of F.

The study of this map is motivated by the *inverse problem* which arises in reflection seismology, oil exploration, ground-penetrating radar, etc. Mathematically, the inverse problem is to determine the coefficient σ by knowing additional boundary value conditions of u. Since the inverse problem is just to invert the functional relation F, we are naturally interested in all the properties of this forward map.

To understand the problem, let us look at a simple exploration seismology experiment: Near the surface of the earth, a seismic source is fired at some point (point source). The seismic waves propagate into the earth. Since the earth's structure varies (as do its physical properties) part of the energy of the wave will be reflected back to the surface and can be measured. The inverse problem is to deduce the interior properties of the earth from the recorded data.

A simple model of this reflection seismic inverse problem in this context is: given data $F_{data}(x', t)$, find a coefficient $\sigma(x)$ so that

$$F(\sigma) = F_{data}$$

or perhaps minimizing the error $(F_{data} - F(\sigma))$ in some norm.

A natural problem of mathematical and physical importance is to pursue the *right* models so that the reflected waves they generate carry sufficient information for determining the physical properties of the medium. By the theory of geometric optics, the models which are too smooth (*i.e.*, the coefficients σ and c are smooth) on the wavelength scale do not generate reflected waves. On the other hand, no energy penetrates extremely oscillatory media, hence models that are too rough generate no reflected waves.

Another important reason that one wants to work with nonsmooth models comes from a computational point of view. It is clear that to solve inverse problems numerically requires efficient minimization algorithms. By far, the most efficient minimization algorithms are Newton-type algorithms. According to the infinite dimensional optimization theory (see e.g. Kantorovich and Akilov [14]), in order to formulate any *effective* convergent Newton-type algorithm, one has to study the problem in a Banach space. Moreover, dealing with minimization problems, the best available results are perhaps those for Hilbert spaces. Note that even though C^∞-topology induces countable semi-norms, it is not a Banach space. At this point, we do not know any effective convergent Newton-type minimization algorithm in a non-Banach space. Besides, it is natural to use the weakest norm and the biggest possible space of models.

When the spatial dimension is one or c and σ depend only on x_n (layered problem) there is a large literature available. For a similar problem in which the medium was assumed to be excited by an impulsive load on the surface $\{x_n = 0\}$ instead of point sources, the properties of the forward map have been studied fairly satisfactorily by Symes and others (see Symes [23] for references). It was shown by Symes that, for the constant wave speed case, the forward map defines a C^1-*diffeomorphism* between open sets in certain Hilbert spaces by applying the method of geometrical optics together with energy estimates.

When the spatial dimension $n > 1$ and c, σ depend on all space variables (nonlayered problem), very little is known in mathematics. Symes [21, 22], Sacks and Symes [19], Rakesh [17], and Sun [20] have some partial results. The difficulties are essentially due to the ill-posed nature of the timelike hyperbolic Cauchy problem and the presence of nonsmooth coefficients. For the one dimensional wave equation, both coordinate directions are spacelike, which indicates that the problem is hyperbolic with respect to both directions. Apparently, this is not the case when the spatial dimension is larger than one.

Rakesh in [17] looked at a related linearized velocity inversion problem with constant density and point sources. Assuming smooth background velocity, he obtained some results on both upper and lower bounds for the linearized forward map. The essential observation in Rakesh's work is that DF is a Fourier integral operator (see also Beylkin [7]). Unfortunately, the calculus of Fourier integral operators employed in Rakesh's work is not applicable to the nonsmooth reference velocity case since the linearized forward map is a Fourier integral operator only when the reference velocity is smooth.

In [21], Symes gave a pair of examples, based on the geometric optics construction, which show that both $DF(1)$ and $DF(1)^{-1}$ are unbounded for a slightly different problem. As the examples show, within the Sobolev scales no strengthening or weakening of topologies of the domain and range can make both DF and DF^{-1} bounded. This fact also implies a strategy of regularization: Change the topology in the domain so that DF becomes bounded, then ask for optimal regularization of DF^{-1} in the sense of best possible lower bound estimate for DF. In both examples of Symes, the unboundedness was caused by rapid oscillation of σ in the x'-direction or the tangential directions, hence the problem is actually "partially well-posed", i.e., only more smoothness of the coefficients in tangential directions (essentially grazing ray directions) will be required to cure the difficulty. This might be the main reason the anisotropic Sobolev spaces $H^{m,s}(\mathbb{R}^n)$ or Hörmander spaces, were introduced in [19] and [20].

In Theorem 4.1 of [19] Sacks and Symes showed by using the method of sideways energy estimates that for a linearized density determination problem with constant velocity and plane wave sources, DF is bounded from $H^{1,1}$ to H^1, provided the reference coefficient is in $H^{1,s}$ for some $s > n+2$. They also proved the injectivity of DF. However, as they pointed out, the lower bound for DF was not that satisfactory. Our techniques and results are quite different from theirs. We intend to assure the optimal regularity of the timelike trace under weaker hypotheses.

There remains an extremely important issue to be addressed, namely,

What is an appropriate space for the domain of DF ?

In 1983, Symes suggested that microlocal restrictions on the coefficients might regularize the inverse problem (see [22] and [24]). In some sense, this was confirmed by Bao and Symes [2] where we were able to prove a trace theorem for the solutions of general linear *p.d.e.* with smooth coefficients. Roughly speaking, our theorem asserts that the solution will belong to H^s along a codimension one hypersurface if it belongs to H^s in a neighborhood of the hypersurface and to H^{s+1} microlocally in those directions where the *p.d.e.* is not microlocally strictly hyperbolic. Note that we gained back the half derivative from the standard trace theorem. In a recent paper [3], we proved a similar time like trace regularity result for a second order hyperbolic equation with nonsmooth coefficients. It is obvious that the presence of nonsmooth coefficients will introduce new singularities to the solutions so that only limited initial regularity can be propagated. A crucial step in [3] was to develop an extended Beals-Reed theorem (Theorem 1 in [6]) on propagation of singularities.

The main result of this paper is a boundedness theorem for the linearized forward map $DF(\sigma_0)$ for the (sufficiently regular) nonsmooth σ_0. The main ingredients of our proof are the method of energy estimates, a regularity study of the fundamental solution, results on propagation of singularities, several trace regularity results, and a useful dual technique.

The plan of this paper is as follows. In Section 2, a regularity theorem for the solution of the model problem is established by applying the method of progressing wave expansions inside the characteristic surface. A simple energy identity plays a crucial role in our analysis: It indicates that the regularity result can be established by analyzing the regularity of the transport equations.

Section 3 is devoted to the proof of our main theorem. A crucial step is to analyze the propagation of regularity for solution of a problem dual to the linearized problem. In this process, a microlocal version of the classical trace theorem is introduced. It is also important to derive an estimate out of the result on propagation of singularities.

Notation. Throughout this paper, the reader is assumed to be familiar with the basic calculus of *Pseudodifferential Operators* (" ψ.*d.o.* ") as stated in Taylor [26] or Nirenberg [16]. A classical ψ.*d.o.* P of order m is denoted as $P \in OPS^m$ with its symbol $p \in S^m$. $ES(P)$ stands for *the essential support* of operator P. $WF(u)$ denotes *the wave front set* of a distribution u. H^s is the standard L^2-type Sobolev space and H^s_{loc} means a local Sobolev space. $\langle\xi\rangle$ means $(1+|\xi|^2)^{1/2}$. For a nice discussion on microlocal Sobolev spaces

$H^s \cap H^r_{m\ell}(x_0, \xi_0)$, we refer the reader to Beals [5], see also Rauch [18]. For simplicity, C serves as a generalized positive constant the precise value of which is not needed.

Warning. When the reference density σ_0 is smooth, most of the regularity results for the forward map in this work will follow more easily from the calculus of *Fourier Integral Operators.* For a standard text on *F. I. O.* we refer to Duistermaat [10] or Hörmander [12]. However, this technique fails with the appearance of the nonsmooth reference density, an assumption important in this work.

2 Regularity of Fundamental Solution

Since the excess pressure u in the model equation is in fact the fundamental solution, in order to study the regularity of the forward map, the regularity of the fundamental solution must be understood. It is evident that the real obstacle here is the singular right-hand side so that none of the propagation of singularity results could be applied to handle it directly. A natural way to cure this difficulty is by employing the Hadamard theory of progressing wave expansion. We refer the reader to Courant and Hilbert [9] or Friedlander [11] for a detail study on the method of progressing wave expansions. According to Hadamard's construction, the fundamental solution may be represented as a sum of the principal part and remainder. One can then study the remainder by the Beals-Reed type propagation of singularity theorem. However, a great drawback of this idea is that additional regularity is needed to regularize the remainder term. In this section, taking the special structure of the model problem into account, we shall modify the above straightforward idea by introducing an energy identity. The advantage of this technique is that with the energy identity, we can essentially get rid of the remainder term in the expansion; therefore a refined regularity result should be expected.

In order to get the regularity for the fundamental solution, it is also crucial to study the transport equations, where with nonsmooth coefficients, the Rauch-type results will be demanded.

2.1 Energy identity

Consider a problem obtained by integrating the model problem in the time variable,

$$\begin{aligned} &(\Box - \nabla\sigma_0 \cdot \nabla)v_0 = \delta^{-\frac{n-1}{2}}(t)\delta(x)\,, \quad (x,t) \in \mathbb{R}^{n+1} \\ &v_0 = 0\,, \quad t < 0\,. \end{aligned} \tag{2.1}$$

Hadamard's construction leads to the progressing wave expansion for v_0,

$$v_0 = \sum_{k=0}^{s} b_k S_k(t - \tau(x)) + R_{v_0}(x,t) \tag{2.2}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$ $(k \geq 1)$, and R_{v_0} vanishes at $t = \tau(x)$. Moreover $\{b_k\}$ solve the transport equations, for $k = 1, \cdots, s$,

$$2\nabla\tau \cdot \nabla b_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)b_0 = 0 \tag{2.3}$$

$$2\nabla\tau \cdot \nabla b_k + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)b_k = \Delta b_{k-1} + \nabla\sigma_0 \cdot \nabla b_{k-1}\,. \tag{2.4}$$

Since the boundedness of the energy norm will naturally lead to the regularity, we attempt to bound the energy norm by recalling an energy identity stated in Symes [24].

Denote

$$B_T = \{x : \tau(x) \le T\}\ ,\quad C_T = \{(x,t) : t = \tau(x) \le T\}\ .$$

We can then introduce an energy identity for the solution of the wave equation.

Proposition 2.1 (Energy Identity) *Suppose w solves the inhomogeneous wave equation*

$$\begin{aligned} (\Box - \nabla\sigma\cdot\nabla)w = f\ , &\quad (x,t) \in \mathbb{R}^{n+1} \\ w = 0\ , &\quad t < 0\ . \end{aligned} \tag{2.5}$$

Define

$$E_T(t) = \int_{B_T} dxe^{\sigma}(|\frac{\partial w}{\partial t}|^2 + |\nabla w|^2)\ .$$

Then the following identity holds

$$E_T(t) = \int_{C_T} dxe^{\sigma}|\frac{\partial w}{\partial t}\nabla\tau + \nabla w|^2 + \int\int_{B_T\times[0,t]} dxdte^{\sigma}fw_t\ . \tag{2.6}$$

Proof. We shall assume that σ, f, w are smooth enough, and w has compact support in x for each t. The equation (2.5) may be rewritten as

$$e^{\sigma}\partial_t^2 w - \nabla\cdot(e^{\sigma}\nabla)w = e^{\sigma}f\ .$$

Multiply both sides by w_t and integrate over $B_T \times [0,t]$,

$$\int\int_{B_T\times[0,t]} dxdte^{\sigma}fw_t = \int\int_{B_T\times[0,t]} dxdt\{\frac{\partial}{2\partial_t}e^{\sigma}(|\frac{\partial w}{\partial t}|^2 + |\nabla w|^2) - \nabla\cdot(e^{\sigma}\nabla w w_t)\}\ .$$

Integration by parts (divergence theorem) yields

$$\int\int_{B_T\times[0,t]} dxdte^{\sigma}fw_t = E_T(t) - \int_{C_T} dxe^{\sigma}|\frac{\partial w}{\partial t}\nabla\tau + \nabla w|^2\ .$$

□

Remarks on the energy identity.

(1) Applying Proposition 2.1 to v_0, the remainder term is eliminated due to the fact that $R_{v_0} = 0$ on C_T. More interestingly, both the tangential and normal derivatives of v_0 are determined by the transport equations.

(2) After a simple calculation, we can deduce from (2.2) that

$$\begin{aligned} \nabla v_0|_{t\to\tau(x)+} &= \nabla b_0 - b_1\nabla\tau \\ \frac{\partial v_0}{\partial t}|_{t\to\tau(x)+} &= b_1\ . \end{aligned}$$

Therefore
$$(\frac{\partial v_0}{\partial t}\nabla\tau + \nabla v_0)|_{t\to\tau(x)^+} = \nabla b_0 \ ,$$
where the term b_1 is killed due to a cancellation. In fact this is true in general: Given P a differential operator with constant coefficients of order k, it is easy to show that b_k does not appear in
$$(\frac{\partial P v_0}{\partial t}\nabla\tau + \nabla P v_0)|_{t\to\tau(x)^+} \ ;$$
the leading term is ∇b_{k-1}.

With Proposition 2.1, one can then examine the regularity of v_0 in (2.1) in terms of the regularity of the solutions to the corresponding transport equations.

Corollary 2.1 *The solution v_0 of (2.1) belongs to H^l inside $\{t = \tau(x)\}$ if and only if $b_k \in H^{l-k}$, where b_k solves the k-th transport equation of (2.3), (2.4) and $k = 0, \cdots, l-1$.*

Proof. From the above energy identity as well as the remarks it is obvious to show that the L_2-norm of $\partial_t^k v_0$ can be bounded by the H^{k-i}-norm of b_i for $i = 1, \cdots, k-1$. Hence it is sufficient to consider higher order x-derivatives. But this is not difficult either. Since v_0 solves the differential equation, it is easy to see that the x-derivatives of v_0 solve some inhomogeneous equations. Then the above energy identity and a use of Gronwall's inequality will lead to the desired estimates. □

2.2 Regularity of fundamental solution

In order to establish a regularity result with the presence of nonsmooth coefficients, we need the following results. The first was originally established by Bony [8] and was extended by Meyer [15]. See also Beals [4] for a different proof.

Proposition 2.2 *Suppose that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, $u \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$, $n/2 < s \le r \le 2s - n/2$, and $g \in C^\infty$, then*
$$g(x, u) \in H^s \cap H^r_{m\ell}(x_0, \xi_0) \ .$$

We also need a Gårding's type inequality concerning the microlocal ellipticity. See Bao [1] Lemma 3.3 for the proof.

Lemma 2.1 *Assume that $Q_1 \in OPS^{m_1}$, $Q_2 \in OPS^{m_2}$, with $m_1, m_2 \in \mathbb{R}$. Furthermore assume that Q_2 is elliptic on $ES(Q_1)$. Then for any $r \in \mathbb{R}$, Ω and Ω' two open bounded sets of $\mathbb{R}^n$ with $\Omega \subset\subset \Omega'$, and $u \in C_0^\infty(\Omega)$,*
$$||Q_1 u||_{s,\Omega} \le C||Q_2 u||_{s+m_1-m_2,\Omega'} + C||u||_{r,\Omega'} \ .$$

Observe that all the transport equations in (2.3)-(2.4) have the same principal part $2\nabla\tau\cdot\nabla$ which is a smooth vector field. Therefore in order to understand the regularity of the solutions to (2.3)-(2.4) it is essential to study the properties of this smooth vector field.

Introducing polar coordinates, we then get $\nabla\tau\cdot\nabla = \frac{\partial}{\partial\lambda}$ $(\lambda = |x|)$. Thus, λ may be treated as the "time" variable for a standard hyperbolic problem. The equation may be expressed under the polar coordinates as

$$\frac{du}{d\lambda} = f\,. \tag{2.7}$$

Lemma 2.2 *Let V be the smooth vector field $\nabla\tau\cdot\nabla$. Suppose that u is smooth and $supp(u) \subset \{\lambda > \delta\}$, for $\forall\delta > 0$ small. Then there exists a $\psi.d.o.$ Q of order zero such that Q is elliptic on $Char(V)$ and $[V,Q] \in OPS^{-\infty}$. Moreover, for $s \in \mathbb{R}$, the inequalities*

$$\begin{aligned} \|\phi u\|_{s,\Omega} &\leq C\|\tilde{\phi}QVu\|_{s,\Omega'} + C\|\tilde{\phi}Vu\|_{s-1,\Omega'} + C\|\tilde{\phi}u\|_{r,\Omega'} & (2.8)\\ \|Qu\|_{s,\Omega} &\leq C\|QVu\|_{s,\Omega} + C\|u\|_{r,\Omega} & (2.9)\end{aligned}$$

hold for any $r \in \mathbb{R}$, where $\phi \in C_0^\infty(\Omega)$ and $\tilde{\phi} \in C_0^\infty(\Omega')$, $\Omega = K \times (\delta^-, T)$, $\Omega \subset\subset \Omega' \subset \{\lambda > \delta^-\}$ with $K \subset \mathbb{R}^{n-1}$, Ω and Ω' are sufficiently big open bounded sets, and $\tilde{\phi} > 0$ on $supp(\phi)$.

Proof. The existence of operator Q follows from Nirenberg's construction which appeared in the proof of Theorem 6 in [16], together with a local compactness argument. The assumption on Q implies that it is elliptic on γ, a small conic neighborhood of $Char(V)$. Thus, one may construct another $\psi.d.o.$ R of order zero which has the properties:

- $R + Q$ is elliptic and
- $ES(R) \cap \gamma = \emptyset$.

Then Gårding's inequality (see *f.g.*, Taylor [26]) gives

$$\begin{aligned} \|\phi u\|_{s,\Omega} &\leq C\|(R+Q)\phi u\|_{s,\Omega} + C\|\phi u\|_{r,\Omega}\\ &\leq C\|R\phi u\|_{s,\Omega} + C\|Q\phi u\|_{s,\Omega} + C\|\phi u\|_{r,\Omega} \end{aligned} \tag{2.10}$$

for any $r \in \mathbb{R}$.

Since V is elliptic on $\gamma^c \supset ES(R)$, the Gårding's type result Lemma 2.1 yields that for a bounded open set Ω_1 with $\Omega \subset\subset \Omega_1$

$$\|R\phi u\|_{s,\Omega} \leq C\|V\phi u\|_{s-1,\Omega_1} + C\|\phi u\|_{r,\Omega_1}\,; \tag{2.11}$$

hence

$$\begin{aligned} \|\phi u\|_{s,\Omega} &\leq C\|V\phi u\|_{s-1,\Omega_1} + C\|Q\phi u\|_{s,\Omega_1} + C\|\phi u\|_{r,\Omega_1}\\ &\leq C\|\phi Vu\|_{s-1,\Omega_1} + C\|\phi Qu\|_{s,\Omega_1} + C\|\phi_1 u\|_{s-1,\Omega_1} + C\|\phi u\|_{r,\Omega_1}\,, \end{aligned}$$

with $\phi_1 \in C_0^\infty(\Omega_1)$, and $\phi_1 > 0$ on $supp(\phi)$. Now we may apply a bootstrap argument. In fact, same analysis leads to

$$\begin{aligned} \|\phi_i u\|_{s-i,\Omega_i} \leq\ & C\|\phi_i Vu\|_{s-i-1,\Omega_{i+1}} + C\|\phi_i Qu\|_{s-i,\Omega_{i+1}} + C\|\phi_{i+1}u\|_{s-i-1,\Omega_{i+1}}\\ &+ C\|\phi_i u\|_{r,\Omega_{i+1}}\,, \end{aligned} \tag{2.12}$$

where $\phi_i \in C_0^\infty(\Omega_i)$, Ω_i bounded open sets, and $\phi_{i+1} > 0$ on $supp(\phi_i)$, $i = 1, 2, \cdots$. Therefore, a simple calculation yields

$$||\phi u||_{s,\Omega} \le C[||\tilde{\phi} V u||_{s-1,\Omega'} + ||\tilde{\phi} Q u||_{s,\Omega'} + ||\tilde{\phi} u||_{r,\Omega'}] \, . \tag{2.13}$$

Thus it suffices to study the term $||Qu||_{s,\Omega'}$. Observe that $Qu \in C^\infty(K)$ solves

$$VQu = QVu + [V, Q]u$$

which is a first order equation. That is,

$$\frac{dQu}{d\lambda} = QVu + [V, Q]u \, . \tag{2.14}$$

Moreover, since $supp(u) \subset \{\lambda > \delta\}$, the pseudolocal property of Q yields that

$$||(Qu)(\cdot, \delta)||_{s,K} \le C||u||_{r,\Omega} \, .$$

Hence the method of hyperbolic energy estimates in Taylor [26] pages 73-75 may be applied to (2.14) and leads to a simple estimate

$$||Qu(\cdot, \lambda)||^2_{s,K} \le C||u||^2_{r,\Omega} + C\int_\delta^\lambda [||QVu(\cdot, \lambda)||^2_{s,K} + ||[V, Q]u(\cdot, \lambda)||^2_{s,K}] d\lambda \, . \tag{2.15}$$

or

$$\begin{aligned} ||Qu(\cdot, \lambda)||^2_{s,K} &\le C||u||^2_{r,\Omega} + C||QVu||^2_{s,\Omega} + C||[V, Q]u||^2_{s,\Omega} \\ &\le C||QVu||^2_{s,\Omega} + C||u||^2_{r,\Omega} \, . \end{aligned} \tag{2.16}$$

Here we have used the fact that $[V, Q]$ is a smoothing operator in getting the second estimate. The estimate (2.9) follows from differentiating the differential equations and estimates which are similar to (2.15).

Substituting the estimate (2.9) to (2.13), we eventually obtain that

$$||\phi u||_{s,\Omega} \le C||\tilde{\phi} V u||_{s-1,\Omega'} + C||\tilde{\phi} Q V u||_{s,\Omega'} + C||\tilde{\phi} u||_{r,\Omega'} \, ,$$

which completes our proof. □

Until now, we have only considered the principal part of the transport equations. Fortunately, our next proposition implies that the lower order terms may actually be absorbed by the principal part, hence the whole analysis can go through.

Proposition 2.3 *Assume that w, q solve*

$$Vw = f \text{ and } Vq = a \, , \tag{2.17}$$

where again V denotes $\nabla\tau \cdot \nabla$. Then $\tilde{w} = we^{-q}$ solves the equation

$$V\tilde{w} + a\tilde{w} = fe^{-q} \, . \tag{2.18}$$

Proof. Substituting $\tilde{w} = we^{-q}$ to the left-hand side of (2.18), one has by chain rule

$$V\tilde{w} + a\tilde{w} = Vwe^{-q} + (-Vq + a)we^{-q} .$$

Hence the assumptions in (2.17) verify the equation (2.18). □

Remark. We want to make the following observation: In the transport equations (2.3) and (2.4),

$$q = \sigma_0/2 + q_0$$

where q_0 solves equation $Vq_0 = \Delta\tau/2$. Thus away from the origin, q is nothing more than a smooth perturbation of $\sigma_0/2$.

With the above preparations, we are now ready to state and prove the main result of this section.

Theorem 2.1 *Suppose that $\sigma_0 \in H^s \cap H^{2l-1}_{m\ell}(Char(\nabla\tau \cdot \nabla))$ with $l + n/4 < s < 2l - 1$. Then for $\{b_k\}$ solving the transport equations (2.3) and (2.4)*

$$v_0 = \sum_{k=0}^{s} b_k S_k(t - \tau(x)) + R_s \quad \textit{and } R_s \in H^l(U) ,$$

where $U = \{(x,t) : \ x \in \Omega \ , \ t \in [0,T] \ and\, t \geq \tau(x)\}$ is a compact set in $\mathbb{R}^{n+1}$.

Proof. By Corollary 2.1, it suffices to show that $b_k \in H^{l-k}(\Omega)$, where b_k is the solution of the k-th transport equation of (2.3) and (2.4), for $k = 0, \cdots, l-1$.

We once again introduce a function $q = \sigma_0/2 + q_0$ with $\nabla\tau \cdot \nabla q_0 = \Delta\tau/2$. Then according to Proposition 2.3, the transport equations (2.3), (2.4) may be transformed to equations

$$\nabla\tau \cdot \nabla b_0 e^q \;=\; 0 \tag{2.19}$$
$$\nabla\tau \cdot \nabla b_k e^q \;=\; (\Delta b_{k-1}/2 + \nabla\sigma_0 \cdot \nabla b_{k-1}/2)e^q \, , \tag{2.20}$$

for $k = 1, \cdots, l-1$. Recall that these transport equations are hyperbolic along the λ direction ($\lambda = |x|$). Moreover, since v_0 is the fundamental solution, the Hadamard construction yields that $\{b_k\}$ $(k = 0, \cdots, l-1)$ are constant near $x = 0$ (or $\lambda = 0$).

From equation (2.19), the assumptions on σ_0 and l clearly indicate that $||b_0||_{l,\Omega} \leq C||\sigma_0||_{l,\Omega}$.

Since $\nabla\sigma_0 \in H^{s-1} \cap H^{2l-2}_{m\ell}(\gamma)$, the assumptions imply that $s-1$ and $2l-2$ satisfy Rauch's condition, *i.e.*,

$$n/2 < s - 1 \leq 2l - 2 < 2(s-1) - n/2 ,$$

hence Proposition 2.3 and the extended Rauch's lemma guarantee that all of the operations involving $\nabla\sigma_0$ may be performed.

To simplify the arguments, we shall use Q_0 to represent all $\psi.d.o.$ of order zero whose essential supports are close to each other and possess the properties of Q in Lemma 2.2.

Therefore by using (2.8) and (2.9) of Lemma 2.2 several times, after some similar simple calculations, one can write down the following inequalities,

$$\begin{aligned} ||\phi b_1 e^q||_{l-1,\Omega} &\leq C||\phi_1 Q_0 b_0||_{l+1,\Omega'} + C||\phi_1 b_0||_{l,\Omega'} + C||\phi_1\sigma_0||_{r,\Omega'} \\ &\cdots \\ ||\phi b_k e^q||_{l-k,\Omega} &\leq C||\phi_1 Q_0 b_0||_{l+k,\Omega'} + C||\phi_1 b_0||_{l,\Omega'} + C||\phi_1\sigma_0||_{r,\Omega'} , \end{aligned}$$

where $k = 1, \cdots, l-1$, $\phi \in C_0^\infty(\Omega)$, $\phi_1 \in C_0^\infty(\Omega')$ with $\Omega \subset\subset \Omega'$, r is any real number, and C depends at most on $||\phi_1\sigma_0||^{Q_0}_{s,2l-1,\Omega'}$. Knowing the regularity of b_0, Lemma 2.1 then completes the proof. □

We want to make some comments on Theorem 2.1. It is unpleasant to have extra $n/4$-order derivatives on σ_0 in the statement of the theorem. This defect cannot be avoided because Rauch's condition is necessary to get the conclusion of Proposition 2.2. At this point, we do not know how to relax the hypothesis as long as the Rauch-type results are employed.

It is known that in their applications to nonlinear wave equations, most of the results based on Rauch's lemma (or the method of Fourier analysis) are limited to relatively weak singularities. This work exhibits that to some extent, strong singularities appearing in the linear wave equation (*e.g.* the fundamental solution) can also be tackled by this Fourier analysis method with the help of a simple energy identity and the progressing wave expansion. The relation between the coefficients and solution with strong singularities remains to be fully understood, especially when the coefficients are less regular.

3 Upper Bound for Linearized Forward Map

Our goal in this section is to determine the appropriate hypotheses under which $DF(\sigma_0)$, the linearization of F about a reference state σ_0, is bounded.

Recall the linearized problem corresponding to the reference state (u_0, σ_0), for $(t, x) \in \mathbb{R}^{n+1}$, $x = (x', x_n)$,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\delta u &= \nabla\delta\sigma \cdot \nabla u_0 \\ \delta u = 0 , &\quad t < 0 , \end{aligned} \tag{3.1}$$

where u_0 is the solution of the model problem corresponding to the reference density σ_0. The linearized forward map can be defined as

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\,|_{x_n=0} , \tag{3.2}$$

where $\phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$, and near $\{x_n = 0\}$.

Once again we consider a related problem,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\, v &= \nabla\delta\sigma \cdot \nabla v_0 \\ v = 0 , &\quad t < 0 , \end{aligned} \tag{3.3}$$

where $\delta u = \partial_t^{\frac{n-1}{2}} v$ and v_0 solves

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v_0 &= \delta^{-\frac{n-1}{2}}(t)\delta(x) \\ v_0 = 0 , &\quad t < 0 , \end{aligned} \tag{3.4}$$

Observe that for $l \in \mathbb{R}$,

$$\begin{aligned} ||DF(\sigma_0)\delta\sigma||_l &= ||\,(\phi\delta u)\,|_{x_n=0}||_l \\ &\leq C||(\phi v)\,|_{x_n=0}\,||_{l_1}, \end{aligned} \tag{3.5}$$

where l_1 denotes $l + (n-1)/2$. Thus the real challenge here is to get an appropriate trace regularity estimate for v on a time-like hypersurface $\{x_n = 0\}$.

Throughout this section, we shall always assume that

(A) $supp(\delta\sigma) \subset \{x_n > \epsilon\}$,

for $\epsilon > 0$ small. In some applications, this assumption is realistic, as the density can be measured directly, near the location of receivers (i.e. $x_n = 0$).

3.1 Statement of theorem

We first state the main result of this section then give a brief description about the idea of its proof. The theorem will be proved in the subsections which follow.

Let $\Omega \subset \mathbb{R}^k$ be open and bounded, $\gamma \subset T^*(\Omega)$. A constant C is said to depend on the $H^s \cap H^r_{m\ell}(\gamma)$-norm of $w \in C_0^\infty(\mathbb{R}^k)$ if for any conic neighborhood Γ of γ there exists a $\psi.d.o.$ Q of order zero with $ES(Q) \subseteq \Gamma$ and $q = 1$ on $\gamma \cap \{(x,\xi) : |\xi| > 1\}$ such that C can be bounded in terms of $||w||_{s,\Omega} + ||Qw||_{r,\Omega}$.

Theorem 3.1 *Assume that* $max\{l+n-1, 3+n/2\} < s < 2l+n-2$ *(*$n \geq 2$*),* $\theta = Char(\nabla\tau \cdot \nabla) = \{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$*, and* $K = \{(x,\xi) \in T^*(\mathbb{R}^n), |\xi_n| \leq \epsilon|\xi|\}$*. Assume that* $\sigma_0 \in H^s \cap H^{l+(n+1)/2}_{m\ell}(K) \cap H^{2l+n-2}_{m\ell}(\theta)$*. Then under the assumption (A), the following estimate holds*

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\psi\delta\sigma||_{l+\frac{n-1}{2}} \tag{3.6}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ *and the constant* C *depends on the* $H^s \cap H^{l+(n+1)/2}_{m\ell}(K) \cap H^{2l+n-2}_{m\ell}(\theta)$*-norm of* σ_0 *but is independent of* $\delta\sigma$*.*

An interesting special case of Theorem 3.1 is remarkable because the additional microlocal smoothness along the tangential direction will then be absorbed.

Corollary 3.1 *In addition to the assumptions in the statement of Theorem 3.1, assume that the spatial dimension* $n \geq 3$*. Then under the assumption (A), the same estimate*

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\psi\delta\sigma||_{l+\frac{n-1}{2}}$$

holds, where the constant C *depends on the* $H^s \cap H^{2l+n-2}_{m\ell}(\theta)$*-norm of* σ_0 *but is independent of* $\delta\sigma$*.*

However, it still remains to see whether or not the additional smoothness along the characteristic variety of transport equations can be removed.

Before getting into the details of the proof, let us first make the following general remarks on this theorem:

The estimate (3.6) above has a similar form to a Rakesh's theorem (Theorem 2.5 in [17]). Actually a formal extension of our proof here could lead to an elementary proof of his theorem. On the contrary, the principal tool of Rakesh's proof, calculus of Fourier integral operators, is not available when the reference density is nonsmooth.

Our approach here is based on the method of energy estimates associated with results on propagation of singularities and various trace regularity results. The beauty of the method of energy estimates is that it possesses useful information on various parameters involved in the estimates.

To simplify the proof of Theorem 3.1, *w. l. o. g.*, we shall first assume that $\delta\sigma$ is a smooth function with (sufficiently big) compact supports. We then derive the estimates. It is also important to see that the coefficient σ_0 is smooth is not a necessary assumption for employing all the techniques involved in our proof. The precise smoothness requirement for σ_0 can be determined easily from the dependence of the constants on σ_0 in the estimates.

In order to clarify the ideas, we prove the theorem in the following steps:

- Applying the trace theorem in [3], assumption (A), as well as results on propagation of singularities, the estimate of $||(\phi v)\,|_{x_n=0}\,||_{l_1}$ may be reduced to the estimate of $||\phi_0 v||_{l_1}$.
- We then decompose $\delta\sigma$ into two pieces: $Q_1\delta\sigma$ (good part) and $Q_2\delta\sigma$ (bad part), correspondingly decompose v into $v_1 + v_2$, so that they can be studied separately and then reassembled.
- We show that the good part actually leads to the desired estimate by the same technique used in the preceding section.
- The most difficult part is to show that the trace of the bad part is actually smoother than that of the good part. In order to do so, we introduce a dual problem. We show that it suffices to analyze how the singularities (regularity) of the solution of the dual problem propagate. The main ingredients in this step are an estimate derived from the propagation of singularities theorem and a microlocal version of the classical trace theorem.

Let $\phi \in C_0^\infty$ be supported inside the characteristic surface and the set $\{x_n < \epsilon/2\}$. Multiplying ϕ to both sides of equation (3.3), we have

$$\begin{aligned} \Box\phi v &= \phi\nabla\sigma_0\cdot\nabla v + [\Box,\phi]v \\ v &= 0\ ,\quad t<0\ . \end{aligned} \tag{3.7}$$

Here we have used the fact that according to the assumption (A), ϕ and $\delta\sigma$ have disjoint supports, so that $\phi\nabla\delta\sigma\cdot\nabla v_0 = 0$.

Once again with l_1 we denote $l + (n-1)/2$.

Lemma 3.1 *Assume that $s > 3 + n/2$, $1 \le l_1 \le s$, and v solves problem (3.7) then there is a $\phi_0 \in C_0^\infty$ supported near supp(ϕ) such that the following estimate holds,*

$$||(\phi v)|_{x_n=0}||_{l_1} \le C||\phi_0 v||_{l_1} , \tag{3.8}$$

where C is a constant depending on the $H^s \cap H_{mt}^{l_1+1}(K)$-norm of σ_0, but is independent of $\delta\sigma$.

Proof. This lemma is a direct application of Theorem 3.1 in [3] by taking into account of the fact that ϕ and $\delta\sigma$ have disjoint supports. □

3.2 Regularity of v_1

Construct two ψ.*d.o.* $Q_1, Q_2 \in OPS^0(\mathbb{R}^n)$, such that

- $Q_1 + Q_2 = I$;
- $ES(Q_2)$ is a small conic neighborhood of$\{\nabla\tau \cdot \xi = 0\}$;
- Q_2's symbol $q_2 = 1$ near $\{\nabla\tau \cdot \xi = 0\} \cap \{(x,\xi),\ |\xi| \ge 1\}$.

An immediate consequence of this construction is that for any ψ.*d.o.* $\tilde{Q}$ whose essential support is near $\{\nabla\tau \cdot \xi = 0\}$, the operator $\tilde{Q}Q_1$ is a smoothing operator. Accordingly, by linearity, the solution to (3.3) may also be decomposed into two pieces,

$$v = v_1 + v_2 ,$$

where v_i (for $i = 1, 2$) satisfies

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v_i &= \nabla Q_i\delta\sigma \cdot \nabla v_0 , \quad (x,t) \in \mathbb{R}^{n+1} \\ v_i &= 0 , \quad t < 0 . \end{aligned} \tag{3.9}$$

Therefore, in order to estimate $||\phi_0 v||_{l_1}$, it suffices to estimate $||\phi_0 v_i||_{l_1}$ for $i = 1, 2$. We shall proceed to estimate the two terms separately because of their different natures.

The analysis of v_1's regularity is parallel to that in Section 2. From (3.9), Hadamard's construction again leads to the progressing wave expansion of v_1,

$$v_1 = \sum_{k=0}^{s} a_k S_k(t - \tau(x)) + R_{v_1}(x,t) , \tag{3.10}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$, R_{v_1} vanishes at $t = \tau(x)$, and $\{a_k\}$ solve the transport equations, for $k = 0, \cdots, s-1$,

$$2\nabla\tau \cdot \nabla a_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)a_0 = -b_0\nabla\tau \cdot \nabla Q_1\delta\sigma \tag{3.11}$$

$$2\nabla\tau \cdot \nabla a_{k+1} + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)a_{k+1} = \Delta a_k + \nabla\sigma_0 \cdot \nabla a_k + \nabla Q_1\delta\sigma(\nabla b_k - b_{k+1}\nabla\tau) \tag{3.12}$$

and b_k as defined in subsection 2.1 solves the k-th transport equation (2.4) for v_0.

In order to get the regularity for v_1 we attempt to bound the energy norm by the energy identity Proposition 2.1 stated in subsection 2.1.

Lemma 3.2 *Suppose that $l_1 + n/4 < s < 2l_1 - 1$. Then*

$$||\phi_0 v_1||_{l_1} \leq C||\delta\sigma||_{l_1} \tag{3.13}$$

holds, where constant C depends on the $H^s \cap H^{2l_1-1}_{m\ell}(\theta)$-norm of σ_0, and θ is a small conic neighborhood of $\{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$.

Proof. Since the proof follows the same pattern as in subsection 2.2, we shall only make the following observation: Applying the same ideas as in subsection 2.2, one should expect an estimate of the following form, for a suitable bounded open set $\Omega \subset \mathbb{R}^n$,

$$||\phi_0 v_1||_{l_1} \leq C||Q_1\delta\sigma||_{l_1,\Omega} + C||PQ_1\delta\sigma||_{2l_1,\Omega}$$

where C depends on σ_0, P is a $\psi.d.o.$ of order zero, and $ES(P)$ near $\{\nabla\tau \cdot \xi = 0\}$. However our construction of Q_1 implies that PQ_1 is a smoothing operator which is why we call v_1 the good part. □

3.3 Microlocal version of trace theorem

In order to estimate the term $||\phi_0 v_2||_{l_1}$, a microlocal version of the classical trace theorem is necessary.

The classical trace theorem in Sobolev spaces characterizes the regularity of a distribution restricted to a hypersurface. Dealing with inverse problems, one always has to face a difficult but crucial question: When does the restriction operator commute with another operator of interest? The result in this subsection indicates that a simple microlocal trace theorem, which not only works on the space restriction but also on the phase space restriction (*i.e.* a trace theorem on cotangent bundles), may lead to a way to cure this difficulty. Let K be a conic set in $\mathbb{R}^n$, $i : x \in \mathbb{R}^n \to (x,0) \in \mathbb{R}^{n+1}$. Define a semi-norm: for γ a conic set of $\mathbb{R}^k$ and $u \in C_0^\infty(\mathbb{R}^k)$,

$$|u|_{\gamma,s} = \left(\int_{\xi\in\gamma} d\xi |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}\right)^{1/2}.$$

Then, a proof of the classical trace theorem (see *e.g.* in Taylor [26], pages 20-21) implies the following inequality.

Proposition 3.1 *For $s > 1/2$, $u \in C_0^\infty(\mathbb{R}^{n+1})$,*

$$|i^*u|_{K,s-1/2} \leq C|u|_{K\times\mathbb{R},s}.$$

Thus the map i^* may be extended to be a bounded map from $H^s_{m\ell}(x \times \mathbb{R}, K \times \mathbb{R})$ to $H^{s-1/2}_{m\ell}(x, K)$, provided $s > 1/2$.

Let Π_2 be the projection map to the frequency space (or the second factor). We may reformulate this result in terms of $\psi.d.o.$.

Proposition 3.2 *If P_1 is a ψ.d.o. of order zero in $\mathbb{R}^n$, with $\Pi_2 ES(P_1) \subset K$, then there exists a ψ.d.o. P_2 of order zero in $\mathbb{R}^{n+1}$, and $\Pi_2 ES(P_2) \subset K \times \mathbb{R}$, such that for $s > 1/2$, $u \in C_0^\infty(\Omega)$ with Ω an open bounded subset of $\mathbb{R}^{n+1}$,*

$$||P_1 i^* u||_{s-1/2,\Omega_0} \leq C||P_2 u||_{s,\Omega} ,$$

where i^* again denotes a restriction operator to a codimension one hypersurface and $\Omega_0 = i^*\Omega$.

The above results together with our Gårding's type result Lemma 2.1 yield a microlocal version of trace theorem.

Lemma 3.3 *Assume that E is an elliptic operator of order m in $\mathbb{R}^{n+1} \times K \times \mathbb{R}$, $P \in OPS^0(\mathbb{R}^n)$ and $\Pi_2 ES(P) \subset K$. Then for $s > 1/2$, $u \in C_0^\infty(\Omega)$ where Ω and Ω' are open bounded subsets of $\mathbb{R}^{n+1}$ with $\Omega \subset\subset \Omega'$, and $\Omega_0 = i^*\Omega$,*

$$||Pi^* u||_{s-1/2,\Omega_0} \leq C||Eu||_{s-m,\Omega'} + C||u||_{r,\Omega'}$$

for any $r \in \mathbb{R}$.

3.4 Dual problem

According to our previous trace regularity result (Theorem 3.1 in [3]), under some appropriate hypotheses (in particular, the assumption (A)), bounding $||(\phi v_2)|_{x_n=0}||_{l_1}$ is equivalent to bounding $||\phi_0 v_2||_{l_1}$. Moreover because of a claim which will be proven at the end of this section, it suffices to bound $||\partial_t^{l_1}\phi_0 v_2||$.

Recall that v_2 solves

$$\begin{aligned} \Box v_2 - \nabla\sigma_0 \cdot \nabla v_2 &= \nabla Q_2 \delta\sigma \cdot \nabla v_0 \\ v_2 &= 0 \quad t < 0 . \end{aligned} \tag{3.14}$$

To simplify the arguments on its dual problem, we make use of the symmetric form of (3.14) by introducing $\rho(x) = e^{\sigma_0}$. Then (3.14) becomes

$$\begin{aligned} \Box_1 v_2 = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)]v_2 &= \frac{1}{\rho}\nabla Q_2 \delta\sigma \cdot \nabla v_0 \\ v_2 &= 0 \quad t < 0 . \end{aligned} \tag{3.15}$$

Now let us look at a dual problem to (3.15),

$$\begin{aligned} \Box_1' w = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)]w &= \psi \\ w &= 0 \quad t >> T_1 , \end{aligned} \tag{3.16}$$

where $\psi \in C_0^\infty(\Omega)$ with Ω an open subset of $\{\mathbb{R}^n \times [0, T_1]\} \cap \{|t| > |x|\}$.

Equivalently, we may reformulate (3.16) as

$$\begin{aligned} \Box_1' w = \Box w + \nabla\sigma_0 \cdot \nabla w &= e^{\sigma_0}\psi \\ w &= 0 \quad t >> T_1 . \end{aligned} \tag{3.17}$$

Thus if we can show that for any $\psi \in C_0^\infty(\Omega)$

$$|(\partial_t^{l_1} v_2, \psi)| \le C ||\delta\sigma||_{l_1} ||\psi||_0 , \tag{3.18}$$

then it can be concluded that

$$||\partial_t^{l_1} v_2||_{0,\Omega} \le C ||\delta\sigma||_{l_1} . \tag{3.19}$$

Lemma 3.4 *Suppose that $l_1 + (n-1)/2 < \tilde{s}$ and θ is a small conic neighborhood of $\{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$. Then the estimate (3.19) holds where the constant C depends on the $H^{\tilde{s}} \cap H^{2l_1-1}_{m\ell}(\theta)$−norm of σ_0.*

Proof. Green's identity and integration by parts lead to

$$\begin{aligned}
&(\partial_t^{l_1} v_2, \psi) = (v_2, \Box_1' \partial_t^{l_1} w) \\
&= (\Box_1 v_2, \partial_t^{l_1} w) - \int_{t=\tau(x)} \frac{1}{\rho}[v_2 \frac{\partial}{\partial n} \partial_t^{l_1} w - \partial_t^{l_1} w \frac{\partial}{\partial n} v_2] ds \\
&= (\frac{1}{\rho} \nabla Q_2 \delta\sigma \cdot \nabla v_0, \partial_t^{l_1} w) - \int_{t=\tau(x)} ds \frac{1}{\rho}[v_2(\partial_t^{l_1+1} - \nabla\tau \cdot \nabla \partial_t^{l_1}) w - \partial_t^{l_1} w (\partial_t \\
&\quad -\nabla\tau \cdot \nabla) v_2] .
\end{aligned} \tag{3.20}$$

The first term in (3.20) is easy to handle. Actually, integration by parts and a simple use of Cauchy-Schwarz inequality lead to

$$\begin{aligned}
|(\frac{1}{\rho} \nabla Q_2 \delta\sigma \cdot \nabla v_0, \partial_t^{l_1} w)| &= |(e^{-\sigma_0} \nabla Q_2 \delta\sigma \partial_t^{l_1-1} \nabla v_0, \partial_t w)| \\
&\le C ||e^{-\sigma_0} \nabla Q_2 \delta\sigma \partial_t^{l_1-1} \nabla v_0||_{0,\Omega} ||\partial_t w||_0 .
\end{aligned} \tag{3.21}$$

The energy estimate on w gives $||\partial_t w||_0 \le C ||\psi||_0$. We may apply the generalized Schauder's lemma twice to obtain

$$\begin{aligned}
||e^{-\sigma_0} \nabla Q_2 \delta\sigma \partial_t^{l_1-1} \nabla v_0||_{0,\Omega} &\le C ||\nabla Q_2 \delta\sigma||_{s_0,\Omega} ||e^{-\sigma_0} \partial_t^{l_1-1} \nabla v_0||_{0,\Omega} \\
&\le C ||\delta\sigma||_{s_0+1} ||v_0||_{l_1,\Omega} ,
\end{aligned} \tag{3.22}$$

where $s_0 > n/2$, and $||v_0||_{l_1,\Omega}$ can be handled by Theorem 2.1, provided that $\sigma_0 \in H^s \cap H^{2l_1-1}_{m\ell}(\theta)$ with $l_1 + n/4 < s < 2l_1 - 1$.

Thus it suffices to estimate the last two terms in (3.20). As usual, one may write down the progressing wave expansion for v_2. Actually, assuming that c_i solves the i-th transport equation $(i = 0, 1)$, we have

$$2\nabla\tau \cdot \nabla c_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0) c_0 = -b_0 \nabla\tau \cdot \nabla Q_2 \delta\sigma \tag{3.23}$$

$$2\nabla\tau \cdot \nabla c_1 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0) c_1 = \Delta c_0 + \nabla\sigma_0 \cdot \nabla c_0 + \nabla Q_2 \delta\sigma (\nabla b_0 - b_1 \nabla\tau) . \tag{3.24}$$

Hence to control the last part of (3.20) we only need to analyze

$$I_0 = \int_{t=\tau(x)} c_0 (\partial_t^{l_1+1} - \nabla\tau \cdot \nabla \partial_t^{l_1}) w - c_1 \partial_t^{l_1} w . \tag{3.25}$$

Since Q_2 is symmetric in the sense that $Q_2^* = Q_2$, the Cauchy-Schwarz inequality deduces

$$\begin{aligned} |I_0| &\leq C||\delta\sigma||_0[||Q_2 f(\sigma_0)Tr(\partial_t^{l_1} P_1 w)||_{0,K} + ||Q_2 g(\sigma_0)Tr(\partial_t^{l_1} w)||_{0,K}] \\ &= C||\delta\sigma||_0 I_1 , \end{aligned}$$

with P_1 a first order differential operator (a linear combination of operators ∂_t and $\nabla\tau\cdot\nabla$), $Tr(u) = u|_{t=\tau(x)}$ a restriction (trace) operator, $K = \Omega\cap\{t = \tau(x)\}$, and f, g smooth functions determined by (3.23) and (3.24). It is not difficult to see that f only depends on σ_0, while g involves $\sigma_0, D\sigma_0$ and $\Delta\sigma_0$.

From Lemma 3.3, we know that there is a $\psi.d.o.$ $\tilde{Q}_2$ of order zero whose essential support is contained in a "cylindrical" conic neighborhood of $ES(Q_2)$ along ω-direction, such that $\Pi supp(\tilde{q}_2)$ is near the characteristic surface and $\Pi supp(q_2)\cap supp(\psi) = \emptyset$. That is,

$$|I_1| \leq C||\tilde{Q}_2 f(\sigma_0)P_1 w||_{l_1+1/2,\Omega} + C||\tilde{Q}_2 g(\sigma_0)w||_{l_1+1/2,\Omega} \,. \tag{3.26}$$

Thus an extended Rauch's lemma and the estimates involved in the proof imply that for $l_1 + 1/2 + n/2 < s_0$, $l_1 - 1/2 + n/2 < s_1$,

$$\begin{aligned} ||\tilde{Q}_2 f(\sigma_0)P_1 w||_{l_1+1/2,\Omega} &\leq C_1(||w||_1 + ||Q_0 w||_{l_1+3/2,\Omega}) \\ ||\tilde{Q}_2 g(\sigma_0)w||_{l_1+1/2,\Omega} &\leq C_2(||w||_1 + ||Q_0 w||_{l_1+1/2,\Omega}) \end{aligned}$$

with C_1 and C_2 depending on $||\psi_0\sigma_0||_{s_0}$ and $||\psi_0\sigma_0||_{s_1}$ respectively, $Q_0\in OPS^0$, $ES(Q_0)$ is near $ES(\tilde{Q}_2)$, and $\Pi supp(q_0)\cap supp(\psi) = \emptyset$.

Hence to finish the proof of Lemma 3.4 it is sufficient to show that

$$||Q_0 w||_{l_1+3/2,\Omega} \leq C||\psi||_0 \tag{3.27}$$

which can be proved by applying Lemma 3.5 below. □

3.5 Regularity for solution of the dual problem

A result on propagation of singularities, see Proposition 1.3.3 in Duistermaat [10] or Theorem 8.2.13 in Hörmander [13], demonstrates the relation between the wavefront of the restriction of a distribution and the wavefront set of its own. Applying this result and Hömander's theorem on propagation of singularities, it is easy to see that $Q_2 Tr(\partial_t^{l_1} P_1 w)$ is smooth for smooth σ_0. However the result does not directly lead to any explicit bound. In this subsection, we shall derive the necessary estimates by using a bootstrap argument. Our idea here is motivated by Nirenberg's proof of Hörmander's theorem on propagation of singularities in [16]. In fact, the main purpose of this subsection is to obtain a real estimate out of his proof.

From now on, a constant C is said to be depending on $||\tilde{\psi}\sigma_0||_s$, if for some $\tilde{\psi}\in C_0^\infty(\mathbb{R}^n)$, C depends on $||\tilde{\psi}\sigma_0||_s$.

Lemma 3.5 *There exists an elliptic $\psi.d.o.$ $\tilde{B}$ of order zero, such that $ES(\tilde{B})$ is contained in Cy, a "cylindrical" conic neighborhood of*

$$\{(x,t,\xi,\omega)\in T^*\mathbb{R}^{n+1}\backslash 0,\ t^2 - |x|^2 = 0, \omega = \nabla\tau\cdot\xi\}$$

along ω direction, and the symbol of $\tilde{B}$, $\tilde{b}$ satisfies

$$\Pi supp(\tilde{b}) \cap supp(\psi) = \emptyset \ .$$

Then, for any $k \in \mathbb{R}$,

$$||\tilde{B}w||_{k,\Omega} \leq C_k ||\psi||_0 \ , \tag{3.28}$$

where $k - 2 + n/2 < s$ and the constant C depends on $||\tilde{\psi}\sigma_0||_s$.

The proof follows by showing two propositions below. Proposition 3.3 really gives an estimate based on Nirenberg's proof of Hörmander's theorem. It indicates that an estimate may be formed near any bicharacteristic, hence near the characteristic variety of operator $\Box = \partial_t^2 - \Delta$. We then proceed in Proposition 3.4 to argue that the remaining part of the cylindrical region, where the operator $\Box$ is elliptic, causes no trouble at all. With a concern about the nonsmooth σ_0, it should not be surprising that both propositions require a commutator argument.

Let β be a null bicharacteristic contained in Cy.

Proposition 3.3 *There exists a ψ.d.o. B of order zero such that B is supported in a conic neighborhood of β, B is elliptic near β, and $\Pi supp(b) \cap supp(\psi) = \emptyset$. If, furthermore, $k - 2 + n/2 < s$, then the estimate*

$$||Bw||_{k,\Omega} \leq C_k ||\psi||_0$$

holds with C_k depending on $||\tilde{\psi}\nabla\sigma_0||_s$.

Proof. According to Nirenberg's construction, one can find a ψ*.d.o.* B_0 of order zero with

(1) b_0 supported in a small conic neighborhood of β, B_0 elliptic near β,

(2) $\Pi supp(b_0) \cap supp(\psi) = \emptyset$, and

(3) $[\Box, B_0] \in OPS^0$.

Since w solves (3.17), the method of energy estimates yields

$$||w||_1 \leq C||\psi||_0 \ ,$$

where C is a constant depending on $||\tilde{\psi}\nabla\sigma_0||_{\tilde{s}}$ for $\tilde{s} > n/2$.

Observe that from (3.17),

$$\Box_1' B_0 w = [\Box, B_0]w - [B_0, \nabla\sigma_0 \cdot \nabla]w + B_0 e^{\sigma_0}\psi \ .$$

Since ψ is supported inside the characteristic surface, $\Pi supp(b_0) \cap supp(\psi) = \emptyset$, we have

$$B_0 e^{\sigma_0}\psi = 0 \ .$$

Now energy estimates give

$$||B_0 w||_{2,\Omega} \leq ||[\Box, B_0]w||_{1,\Omega} + ||[B_0, \nabla\sigma_0 \cdot \nabla]w||_{1,\Omega}) \ . \tag{3.29}$$

Since $[\Box, B_0]$ is of order 0,

$$||[\Box, B_0]w||_{1,\Omega} \le C||w||_1 \le C||\psi||_0 .$$

The third term in (3.29) may be estimated by applying the generalized commutator lemma, Lemma 2.4 in [3] and the corresponding estimate. In fact, let $1 + n/2 < s_0$, we then have

$$||[B_0, \nabla\sigma_0 \cdot \nabla w]||_{1,\Omega} \le C||w||_1 \le C||\psi||_0 ,$$

where C depends on $||\tilde{\psi}\nabla\sigma_0||_{s_0}$.

Thus

$$||B_0 w||_{2,\Omega} \le C_0||\psi||_0 , \tag{3.30}$$

with C_0 depending on $||\tilde{\psi}\nabla\sigma_0||_{s_0}$.

Applying Nirenberg's construction once again, we can find a $\psi.d.o.$ B_1 such that $ES(B_1) \subset ES(B_0)$(strictly), B_1 also has properties (1) and (2) above; moreover $[\Box, B_1] \in OPS^{-1}$ and B_0 is elliptic near $ES(B_1)$. From (3.17) and $B_1 e^{\sigma_0}\psi = 0$,

$$\Box_1' B_1 w = [\Box, B_1]w - [B_1, \nabla\sigma_0 \cdot \nabla]w .$$

If we write down the energy estimates, after a simple $\psi.d.o.$ cut-off on B_1, we will find

$$||B_1 w||_{3,\Omega}^2 \le C||w||_1^2 + C||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_{2,\Omega}||B_1 w||_{3,\Omega} ,$$

where $A_1 \in OPS^0$, $ES(B_1) \subset ES(A_1) \subset ES(B_0)$, B_0 is elliptic on $ES(A_1)$, and $a_1 = 1$ on $ES(B_1) \cap \{(x,\xi), |\xi| \ge 1\}$.

Now since $w \in H^1 \cap H^2_{m\ell}(ES(B_0))$, Lemma 2.4 in [3] again implies that $[B_1, \nabla\sigma_0 \cdot \nabla]w \in H^1 \cap H^2_{m\ell}(ES(A_1))$ and

$$||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_{2,\Omega} \le C(||w||_1 + ||A_1 w||_{2,\Omega}) .$$

Here C depends on $||\tilde{\psi}\nabla\sigma_0||_{s_1}$ for $2 + n/2 \le s_1$.

Because of our construction, B_0 is elliptic on $ES(A_1)$; therefore Gårding's type inequality Lemma 2.1 leads to, for any real r and $\Omega \subset\subset \Omega_1$

$$||A_1 w||_{2,\Omega} \le C||B_0 w||_{2,\Omega_1} + C||w||_r \le C||\psi||_0$$

by (3.30).

Therefore we have shown that

$$||B_1 w||_{3,\Omega} \le C_1||\psi||_0 ,$$

where C_1 depends on $||\tilde{\psi}\nabla\sigma_0||_{s_1}$.

We can continue this process by constructing a sequence of $\psi.d.o.$ B_i and A_i ($i = 1, \cdots, k-2$), such that

- B_i has properties (1), (2), $[\Box, B_i] \in OPS^{-i}$,

- $ES(B_{i-1}) \subset ES(A_{i-1}) \subset ES(B_i)$, and
- B_i is elliptic on $ES(A_{i-1})$, $a_{i-1} = 1$ on $ES(B_{i-1}) \cap \{(x,\xi), |\xi| \geq 1\}$,
- Also
$$||B_i w||_{i+2,\Omega} \leq C_i ||\psi||_0 ,$$
where C_i depends on $||\tilde{\psi}\nabla\sigma_0||_{s_i}$ for $i + n/2 < s_i$.

Eventually we conclude by choosing $B = B_{k-2}$ so that, for $k - 2 + n/2 < s$,

$$||Bw||_{k,\Omega} \leq C||\psi||_0$$

with C depending on $||\tilde{\psi}\nabla\sigma_0||_s$. □

Proposition 3.4 *Let P be a ψ.d.o. of order zero with the following properties: The wave operator $\Box$ is elliptic in a small conic neighborhood of $ES(P)$ and $\Pi supp(p) \cap supp(\psi) = \emptyset$. Then*

$$||Pw||_{k,\Omega} \leq C||\psi||_0 ,$$

where C depends on $||\tilde{\psi}\nabla\sigma_0||_q$ for $k - 2 + n/2 < q$.

Proof. The proof is based on the same type of bootstrap arguments as in the proof of last proposition.

Recall (3.17)

$$\Box w + \nabla\sigma_0 \cdot \nabla w = e^{\sigma_0}\psi . \tag{3.31}$$

From the support assumption on p, we see that $Pe^{\sigma_0}\psi = 0$. Hence, by applying P to both sides of (3.31), we find

$$\Box Pw = [\Box, P]w - [P, \nabla\sigma_0 \cdot \nabla]w - \nabla\sigma_0 \cdot \nabla Pw . \tag{3.32}$$

Now since $\Box$ is elliptic in a small conic neighborhood of $ES(P)$, there exists a $\psi.d.o.$ P_0 of order zero, such that $ES(P) \subset ES(P_0)$, P_0 is elliptic near $ES(P)$, and $\Box$ is elliptic in a small conic neighborhood of $ES(P_0)$. From the ellipticity of $P_0\Box$ on $ES(P)$, Proposition 3.3 gives, for any real number r and $\Omega \subset\subset \Omega'$,

$$||Pw||_{k,\Omega} \leq C||P_0\Box Pw||_{k-2,\Omega'} + C||w||_r ,$$

or from (3.32)

$$||Pw||_{k,\Omega} \leq C(||P_0[\Box, P]w||_{k-2,\Omega} + ||P_0[P, \nabla\sigma_0 \cdot \nabla]w||_{k-2,\Omega} + ||P_0\nabla\sigma_0 \cdot \nabla Pw||_{k-2,\Omega}) .$$

Therefore an application of Lemma 2.4 and the generalized Rauch's lemma in [3] yields

$$\begin{aligned} ||Pw||_{k,\Omega} &\leq C_1||P_0 w||_{k-1,\Omega'} + C_2(||w||_1 + ||P_0 w||_{k-2,\Omega}) \\ &\quad + C_3(||w||_1 + ||P_0 w||_{k-1,\Omega}) \\ &\leq C||\psi||_0 + C||P_0 w||_{k-1,\Omega'} . \end{aligned}$$

Here constants C_2 and C_3 depend on $||\tilde{\psi}\nabla\sigma_0||_q$ for $k-2+n/2<q$.

Thus the bootstrap arguments on P_0 will accomplish the proof. $\square$

A combination of Propositions 3.3, 3.4, and Gårding's type Lemma 2.1 assures the existence of an elliptic operator B with properties stated in Lemma 3.5. Now let us look at the cylindrical region: Near the characteristic variety of $\square$, Proposition 3.3 and an extension of Lemma 3.2 may be used; while away from its characteristic set operator $\square_1'$ is microlocally elliptic, hence Proposition 3.4 becomes applicable.

We conclude this section by proving an earlier claim.

Proposition 3.5 *Assume that v_2 solves equation (3.14), $l_1 \in \mathbb{R}$, $l_1 - 3/2 + n/2 < s$. Then the following estimate holds:*

$$||v_2|_{x_n=0}||_{l_1,\Omega_0} \leq C||\partial_t^{l_1} v_2|_{x_n=0}||_{0,\Omega_1} + C||\delta\sigma||_{l_1}\ , \tag{3.33}$$

where Ω_0 and Ω_1 are bounded open sets in $\mathbb{R}^n$ with $\Omega_0 \subset\subset \Omega_1$, and the constant depending on $||\tilde{\psi}\sigma_0||_s$.

Proof. We first construct a *ψ.d.o.* $A \in OPS^0$ such that a, the symbol of A, is equal to one on $|\omega| \geq \epsilon|\xi'|$, for $\xi = (\xi', \xi_n)$, and $ES(A) \subset \{|\omega| \geq \epsilon_0|\xi'|$, with $\epsilon > \epsilon_0\}$. Denote Tr as the restriction operator to $\{x_n = 0\}$; then we have

$$Tr(v_2) = v_2|_{x_n=0} = ATr(v_2) + (I-A)Tr(v_2)$$

or

$$||Tr(v_2)||_{l_1,\Omega_0} \leq ||ATr(v_2)||_{l_1,\Omega_0} + ||(I-A)Tr(v_2)||_{l_1,\Omega_0}\ .$$

Since the operator $\partial_t^{l_1}$ is elliptic on $ES(A)$, a simple use of Lemma 2.1 leads to

$$||ATr(v_2)||_{l_1,\Omega_0} \leq C||\partial_t^{l_1} Tr(v_2)||_{0,\Omega_1} + C||Tr(v_2)||_{r,\Omega_1}$$

for any $r \in \mathbb{R}$.

On the other hand, the microlocal trace theorem implies that there exists a *ψ.d.o.* $\tilde{A}$ of order zero such that $ES(\tilde{A}) \subset$ a cylindrical neighborhood of $\{|\omega| \leq \epsilon|\xi'|\}$ along the ξ_n-direction, and

$$||(I-A)Tr(v_2)||_{l_1,\Omega_0} \leq C||\tilde{A}v_2||_{l_1+1/2,\Omega}$$

where Ω is an open set in $\mathbb{R}^{n+1}$ and its restriction to $\{x_n = 0\}$ contains the set Ω_0.

Therefore similar arguments as in the preceding subsection yield

$$||\tilde{A}v_2||_{l_1+1/2,\Omega} \leq C||\delta\sigma||_{l_1}$$

with the constant C depending on $||\tilde{\psi}\sigma_0||_s$, for $l_1 - 3/2 + n/2 < s$.

Combining the above discussions, we have proved the claim. $\square$

References

[1] G. Bao, Microlocal regularity of an inverse problem for the multidimensional wave equation, Ph.D. thesis, Department of Mathematical Sciences, Rice University, Houston, Texas (1990).

[2] G. Bao and W. Symes, A trace theorem for solutions of linear partial differential equations, Math. Meth. in the Appl. Sci., to appear.

[3] G. Bao and W. Symes, Trace regularity result for a second order hyperbolic equation with nonsmooth coefficients, Technical report 91-1, Department of Mathematical Sciences, Rice University (1991).

[4] M. Beals, Propagation of smoothness for nonlinear second order strictly hyperbolic equations, Proc. of Symp. in Pure Math. 43 (1985), 21-44.

[5] M. Beals, Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems, Birkhäuser, Boston, 1989.

[6] M. Beals and M. Reed, Propagation of singularities for hyperbolic pseudodifferential operators and applications to nonlinear problems, Comm. Pure Appl. Math. 35 (1982), 169-184.

[7] G. Beylkin, Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, J. Math. Phys. 26 (1985), 99-108.

[8] J. -M. Bony, Calcul symbolique et propagation des singularitéaires pour les équations aux dérivées partielles non-linéaires, Ann. Scien. Ecole. Norm. Sup. 14 (1981), 209-246.

[9] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol.2, Wiley-Interscience, New York, 1962.

[10] J. J. Duistermaat, Fourier Integral Operators, Courant Institute Lecture Notes, New York, NY, 1973.

[11] F. G. Friedlander, Sound Pulses, Cambridge University Press, Cambridge, 1958.

[12] L. Hörmander, Fourier integral operators I, Acta Mathematica 127 (1971), 79-183.

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol I, Springer-Verlag, New York, 1983.

[14] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces, Russian ed., Moscow, 1959. Tr. by D. E. Brown, Pergamon, New York, 1964.

[15] Y. Meyer, Régularité des solutions des équations aux derivées partielles nonlinéaires, Sem. Bourbaki, No. 560 (1979-1980).

[16] L. Nirenberg, Lectures on Linear Partial Differential Equations, CBMS Regional Conf. Ser. in Math., No. 17, Amer. Math. Soc., Providence, R.I., 1973.

[17] Rakesh, A linearized inverse problem for the wave equation, Comm. in P.D.E., 13 (1988), 573-601.

[18] J. Rauch, Singularities of solutions to semilinear wave equations, J. Math. Pures Appl. 58 (1979), 299-308.

[19] P. Sacks and W. Symes, Uniqueness and continuous dependence for a multidimensional hyperbolic inverse problems, Comm. in P.D.E., 10 (1985), 635-676.

[20] Z. Sun, Uniqueness theorems for inverse problems in wave propagation and electrostatics, Ph.D. Thesis, Dept. of Math., Univ. of California, Los Angeles, 1987.

[21] W. Symes, A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources, Math. Meth. in the Appl. Sci. 5 (1983), 131-152.

[22] W. Symes, Some aspects of inverse problems in several dimensional wave propagation, Proc. Conference on Inverse Problems, SIAM-AMS Proceedings 14, ed. D. W. McLaughlin, Amer. Math. Soc., Providence, R.I., 1983.

[23] W. Symes, On the relation between coefficient and boundary values for solutions of Webster's horn equation, SIAM J. Math. Anal. 17 (1986), 1400-1420.

[24] W. Symes, Propagation of singularities and some inverse problems in wave propagation, Random Media & Composites, ed. R. V. Kohn and G. W. Milton, SIAM, Philadelphia, 1989, 67-88.

[25] W. Symes, Mathematical foundations of reflected wave imaging, preprint, 1990.

[26] M. Taylor, Pseudo-Differential Operators, Princeton Univ. Press, Princeton, N.J., 1981.