An Empirical Exploration of the Poincaré Model for Hyperbolic Geometry

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Introduction:

Several interactive software packages have been developed which allow students to explore Euclidean geometry. For example, see *The Geometric Supposer* (available from Sunburst) and *The Geometry Sketch Pad* (available from Key Curriculum Press). Because of its graphics capability, the computer offers a high degree of visualization by quickly drawing and measuring geometric figures with a precision that otherwise would require complex drawing instruments, technical skills, and time. These graphics capabilities allow students to explore geometric patterns and theorems not in the usual curriculum. Using these geometry programs, high school students have actually discovered several completely new theorems.[3]

Conspicuously absent from the available geometry software packages is an interactive exploration program for Non-Euclidean geometries. This paper discusses a computer program, called *NonEuclid*, designed to allow students to investigate Hyperbolic Geometry. NonEuclid is available from Rice University, CITI/CRPC, Houston Tx. 77251, Attn.: Danny Powell, or via e-mail softlib@cs.rice.edu.

In the Standard 7 of the National Council of Teachers of Mathematics it is stated for college-intending students to: "Develop an understanding of an axiomatic system through investigating and comparing various geometries" (p. 157).

The truth value of even simple statements like "The base angles of an isosceles triangle are equal" is not obvious in Hyperbolic Geometry. When students try to empirically verify which Euclidean theorems hold in the Hyperbolic Plain they might become better able to differentiate between the concepts of Definition, Postulate, and Theorem.

NonEuclid is designed to be easy to use and fun, while giving students greater insight into Euclidean geometry, and an introduction to Non-Euclidean Geometry. We chose the Poincaré Model for our implementation because it is two-dimensional, bounded, and therefore, easy to represent on a computer screen. NonEuclid runs on Macintosh computers.

The following sections provide:
- a brief explanation and historical description of Non-Euclidean Geometry,
- a description of the Poincaré Model,
- an overview of NonEuclid,
- some empirical results from NonEuclid, and
- a mathematical discussion of some Algorithms used in NonEuclid.

Historical Notes:

In developing his geometry, Euclid (300 BC) formulated five postulates. We will present them here in a revised form given by Proclus (410-485 A.D.) [2]:

P-1 Every two points lie on exactly one line.

P-2 Any line segment with given endpoints may be continued in either direction.

P-3 It is possible to construct a circle with any point as its center and with a radius of any length.

P-4 Any pair of congruent adjacent angles, is congruent to all other pairs of congruent adjacent angles.

P-5 (Parallel Postulate): Given a line L and a point P not on L, there is one and only one line L' which contains P and is parallel to L.

The fifth postulate, called the *Parallel Postulate* always stood apart from the first four. For over 2000 years there were numerous attempts to prove the Parallel Postulate using the first four Postulates. It was not until the nineteenth century that Lobachevski (1793-1856), Bolyai (1777-1855), and Gauss (1802-1860) finally put end to this impossible search. Lobachevski developed theorems using Euclid's first four postulates and the negation of the Parallel Postulate. He expected to eventually "prove" two theorems which contradicted each other. This would imply that negating the Parallel Postulate is inconsistent with the first four postulates - thereby proving the Parallel Postulate. To his surprise, he never obtained a contradiction. Instead, he developed a complete and consistent geometry, a non-Euclidean Geometry. This proved that the fifth postulate could not be derived from the other four. In the early 1900's, Einstein (1878-1955) developed The General Theory of Relativity and made extensive use of Non-Euclidean Geometry.

Hyperbolic Geometry is a Non-Euclidean Geometry based on the first four of Euclid's postulates together with the following variation of the Parallel Postulate.

Hyperbolic Parallel Postulate: Given a line L and a point P not on L, there are at least two lines L' and L'' which contain the point and are parallel to L.

In addition to these postulates, both Euclidean and Hyperbolic geometry require a number of common notions such as "Things which are equal to the same thing are also equal to one another" and "of any three points on a line, exactly one is between the other two." The important point is that all of these common notions are exactly the same for both geometries. In fact, *the only difference between the complete axiomatic formulation of Euclidean Geometry and of Hyperbolic Geometry is the Parallel Postulate*.

The Poincaré model - Definitions and Notation:

To develop the Poincaré model for Hyperbolic Geometry, consider a fixed circle, $\mathcal{C}$, in a Euclidean plane. We assume, without loss of generality, that the radius of $\mathcal{C}$ is 1.

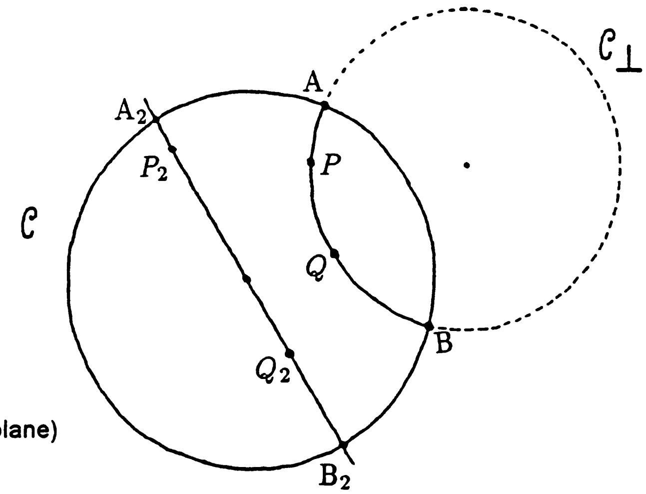


Figure 1 (Lines in the Poincaré plane)

Let $\mathcal{C}_\perp$ be a circle which is orthogonal to circle $\mathcal{C}$. Two circles are orthogonal when their tangents at each intersection point are perpendicular. In the following discussion, P-points, P-lines, etc. are used to identify how points, lines, etc. are defined in the Poincaré model.

P-points: P-points are Euclidean points of the interior of $\mathcal{C}$. Let Ω denote the set of all P-points

P-lines: A P-line is either (1) the intersection of Ω and $\mathcal{C}_\perp$ or (2) the intersection of Ω and a diameter of $\mathcal{C}$.

By drawing a few example P-points it becomes clear that **P-1** is satisfied (Every two points lie on exactly one line). The appendix shows how to find the Euclidean equation of circle $\mathcal{C}_\perp$ determined by any two P-points.

P-2 is also relevant to our definition of P-lines. **P-2** states that any line segment with given endpoints may be continued in either direction. This is satisfied because P-lines form open intervals (P-points may be arbitrarily close to $\mathcal{C}$, but they may not actually be on $\mathcal{C}$).

The Poincaré Model also includes a distance function. Before we define this function, let us consider the minimum set of properties that any reasonable notion of distance ought to satisfy.

D-1 The P-distance between any two P-points P and Q should be a function, d($\overline{PQ}$), which produces a non-negative real number.

D-2 d($\overline{PQ}$) = 0 if and only if $P = Q$

D-3 d($\overline{PQ}$) = d($\overline{QP}$) for every pair of P-points.

D-4 For any three P-points P, Q and R, d($\overline{PR}$) + d($\overline{RQ}$) $\geq$ d($\overline{PQ}$). This states that "a straight line is the shortest distance between two points".

D-5 The distance function needs to satisfy **P-3** which states that it is possible to construct a circle with any point as its center, and with a radius of any length. This Postulate implies that space is continuous, and that for any P-point P, there exists a P-point Q such that d($\overline{PQ}$) is arbitrarily small. This also implies that space is infinite, and that for any P-point P, there exists a P-point Q such that d($\overline{PQ}$) is arbitrarily large.

We now give the Poincaré distance function:

P-distance: Let P and Q denote two P-points. **P-1** tells us that these P-points determine a unique P-line. This P-line will approach $\mathcal{C}$ in two Euclidean points, A and B (notice that A and B are not P-points). Let PA, PB, QA, and QB denote the usual Euclidean distances from P to A, from P to B, etc. The P-distance between P-points P and Q is:

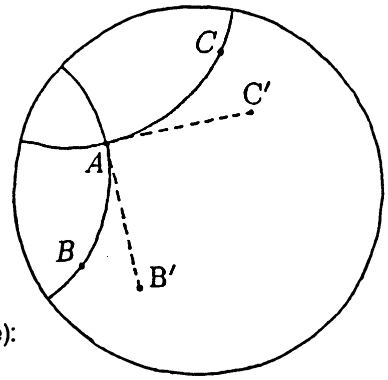
$$d(\overline{PQ}) = \left| \ln \frac{\mathrm{PA}/\mathrm{PB}}{\mathrm{QA}/\mathrm{QB}} \right|.$$

This definition satisfies **D-1** through **D-4** and the postulates of Hyperbolic Geometry. See [1] for a proof. The above formula is presented for completeness. The reader does not need to either visualize or "plug-in" to this formula. The function of NonEuclid is to grind through this dirty work allowing the user to work on the higher level of creating and examining geometric figures of the hyperbolic plane.

An angle in the Poincaré model is formed by intersecting P-lines analogous to the formation of angles in Euclidean Geometry.

P-angle measure Given an angle in the Poincaré model, we form a Euclidean angle by using the two tangent rays (see Figure 2). We define the P-angle measure of $\angle BAC$ to be the Euclidean measure of $\angle$ B'AC'

Figure 2 (Poincaré Angle Measure):



We have explicitly defined points, lines, distance, and angle measure in the Poincaré model. Later we will give an explicit definition of P-area. All of the other objects that will be used in the Poincaré model are defined equivalently to the analogous Euclidean definition. For example, In Euclidean Geometry "parallel lines" are defined as a pair of Euclidean lines which do not intersect. In Hyperbolic Geometry, "P-parallel lines" are defined as a pair of P-lines which do not intersect. This same replacement system gives definitions for P-congruent, P-circles, P-right angles, etc.

Edwin Moise in his book *Elementary Geometry From an Advanced Standpoint* gives a systematic development of the Poincaré model. He shows that the definitions presented are consistent with Euclid's first four postulates and the Hyperbolic Parallel Postulate. His book is quite readable and is highly recommended.

For the remainder of the paper, we will use the following notation:

The Euclidean unit circle $\mathcal{C}$ which contains all P-points will be called the Boundary Circle of the Poincaré Model, or more simply, the *boundary*. The P-point located at the center of the Boundary Circle will be called the *origin*.

Let XY denote the infinite P-line determined by P-points X and Y.
Let $\overline{XY}$ denote the P-line segment with P-endpoints X and Y.
Let $d(\overline{XY})$ denote the P-length of the P-line segment $\overline{XY}$.
Finally, we will drop the P- prefix to Poincaré points, lines, etc.

An Overview of NonEuclid:

Distinguishing between the concepts of Definition, Postulate, and Theorem is critical to understanding the process of a geometric proof. However, many students complete a geometry course without gaining a clear idea of these distinctions. Consider the following statements:

1) A rectangle is a quadrilateral containing four right angles.

2) A rectangle is a parallelogram containing a right angle.

In a typical geometry course we may give students either (1) or (2) as the definition of a rectangle and ask them to prove the other. However, to the student, the "real" definition of a rectangle is a class of shapes which he or she remembers seeing since infancy. The student draws the shape on paper, looks at it and says: "yes, the opposite sides are both parallel and congruent, and yes, it has four right angles - so what are you asking me to do?" It seems to many students like we are just playing word games.

However, when students are given an interactive experience with the Poincaré Model, they will discover that they are able to construct a figure that fulfills definition (2), yet does not have four right angles. Hyperbolic triangles, rhombuses and circles are significantly different from their Euclidean counterparts. It therefore becomes very natural for students to ask what qualities must an object have for it to be called a circle or, what *defines* a circle. Likewise, what qualities are just properties or theorems about circles.

In modern physics and engineering, it is becoming increasingly important to define sets and operations on those sets which may be particular to a given problem. It then becomes important to discover the properties of the new operation (i.e., associative, commutative, distributive, etc.). This is the realm of Abstract Algebra - a course that is often not taught until graduate school.

An interactive exploration of non-Euclidean geometry can offer insight into Abstract Algebra. For example, students working with NonEuclid quickly notice that there are "short" line segments which have greater length then some "long" line segments. This experience will cause students at first to question the validity of the software and ultimately to think of distance in a more abstract sense. They might notice, for example, that the Poincaré Model and the Euclidean plane share the property that a "straight" line is the shortest "distance" between two points. (*Straight* and *distance* are in quotation marks because they do not have the same meanings in both geometries.)

When the user runs NonEuclid, most of the screen is taken up by a Euclidean unit circle which contains the totality of Poincaré points. By selecting menu options, the user can draw and measure Poincaré points, lines, circles, angles, perpendiculars, etc.

Excluding the origin, each Poincaré point, X, has associated with it a unique ordered pair (distance and angle). The distance is the P-distance between X and the origin. The angle measure is between 0° and 360°. This ordered pair we call the *Poincaré Polar Coordinates* of X.

As the cursor is moved around the Poincaré plane, a display shows the Poincaré Polar Coordinates of the cursor position.

Figure 3 shows a snap-shot of the Macintosh screen after 9 points have been plotted and two line segments have been drawn. Whenever a point is plotted, its Poincaré Polar Coordinates are recorded in the scroll column to the right of the graphics display. Notice that point A is at the origin (r=0.00). Notice also that points B, C, D, E, and F are 1, 2, 3, 4, and 5 distance units from the origin. A result of the Poincaré distance formula given in the previous section, is that as a point approaches the boundary, the distance between that point and the origin approaches infinity. On a computer screen, we are limited to a granularity of one pixel. Point Y in figure 3 (r=8.00) is so close to the boundary (in the Euclidean sense) that it appears to be on the boundary; however there are still an infinite number of points and an infinite distance between Y and the boundary.

When we say that C is a distance of 2.00 from the origin we do not assign any units to the distance. Recall that the P-distance is defined in terms of Euclidean distances, and the Euclidean distances are determined by the assumption that the Euclidean radius of the boundary circle is exactly 1.

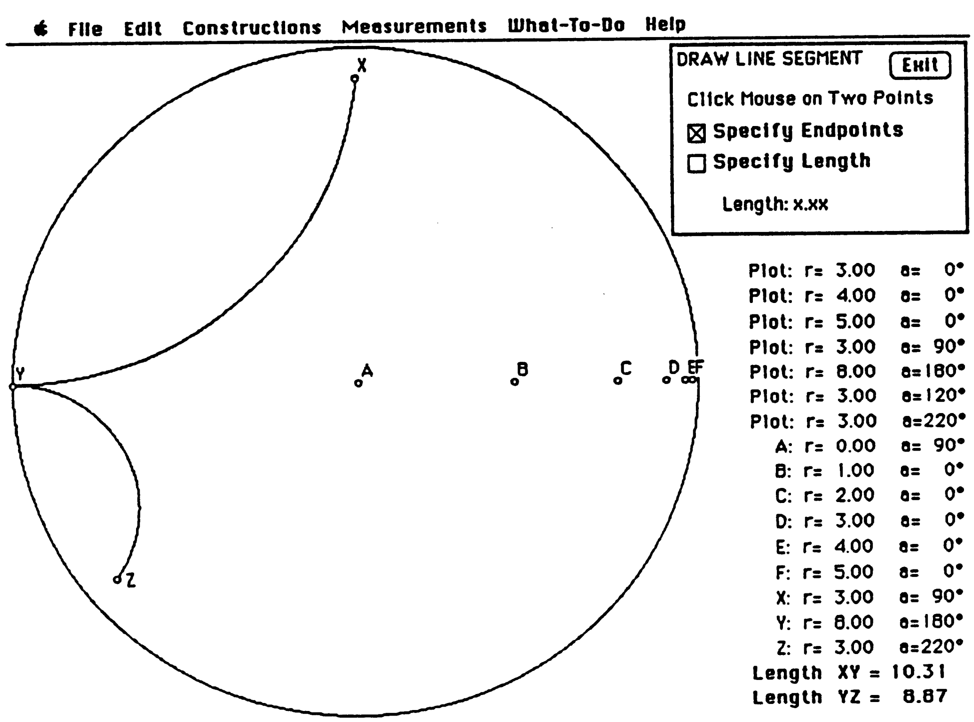


Figure 3 (Snap-shot of NonEuclid on the Mac):

Empirical Exploration:

We can get a better sense of the hyperbolic world by examining figure 4 which shows a tiling of three hundred and sixty congruent right triangles. The triangles are congruent in that the corresponding sides and angles all have equal measure (as measured by the hyperbolic distance and angle functions). Readers who are familiar with the art of M. C. Escher [1902-1972] might recognize figure 4. His "Circle Limit" drawings and woodcuts, show tilings of congruent figures of the Poincaré plane.[5]

Students who work with NonEuclid often expect that figures which are symmetric about the origin would have special properties. For example, it might be believed that the base angles of an isosceles triangle are congruent if and only if the two congruent sides are symmetric about the origin (as in ΔIJK of figure 5). However, exploration shows us that off-center isosceles triangles also have congruent base angles (as in ΔXYZ). While this may baffle the intuition, the proof lies in every high school geometry book. Recall that the only axiomatic difference between Euclidean and Hyperbolic geometry is the Parallel Postulate. Therefore, any proof in Euclidean Geometry which does not use the Parallel Postulate is also a proof in Hyperbolic Geometry. This includes the Isosceles Triangle Theorem, the Angle-Addition Theorem, the Vertical Angle Theorem, the SSS Theorem, and many others. On the other hand, the Euclidean theorems that require the Parallel Postulate will be false in Hyperbolic Geometry, e.g., "the sum of the angles of a triangle equals 180°". Furthermore, neither Euclidean nor Hyperbolic geometry has an absolute "center" or "origin" about which there are special properties. This is very important to the Theory of Relativity.

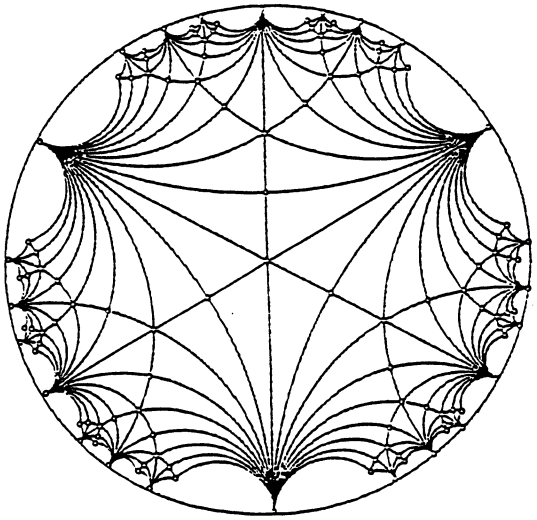


Figure 4 (Tiling the plane)

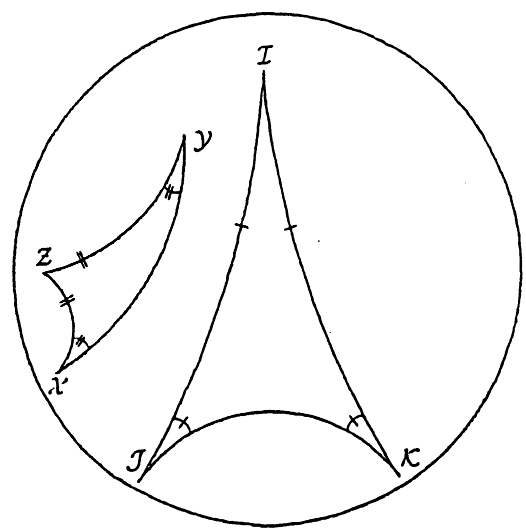


Figure 5 (Isosceles Triangles)

$d(\overline{IJ}) = 6.0$ $m\angle J = 4°$	$d(\overline{ZX}) = 2.0$ $m\angle X = 15°$
$d(\overline{IK}) = 6.0$ $m\angle K = 4°$	$d(\overline{ZY}) = 2.0$ $m\angle Y = 15°$
$d(\overline{JK}) = 6.7$ $m\angle I = 8°$	$d(\overline{XY}) = 3.3$ $m\angle Z = 40°$

Notice that two lines KI and KJ are both parallel to XY yet they intersect each other at K. Therefore, the Euclidean Parallel Postulate is *not* satisfied.

As an illustration of the applicability of some Euclidean theorems to Hyperbolic Geometry, we redraw isosceles ΔXYZ of figure 5 with altitude $\overline{ZP}$. Measurements show that this altitude bisects both the base $\overline{XY}$ and the vertex Z:

$d(\overline{XP}) = d(\overline{PY}) = 1.95$

$m\angle XZP = m\angle YZP = 20°$

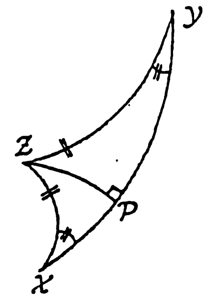


Figure 6 (Altitude of an Isosceles Triangle):

Figure 7 shows three separate clusters of line segments. Cluster A is composed of 36 line segments all of which have length equal to 1 unit. The 36 segments of A also share a common endpoint. Therefore, this cluster can be thought of as forming radii of a circle with center at the common endpoint. Cluster B is of similar construction and shows 36 radii having length equal to 3. Any pair of adjacent radii in either circle A or B form a 10° angle. C is the "largest" circle of the three. C has a radius of 4 and any pair of adjacent radii form a 1° angle.

It is interesting to see that the set of P-points equal distant from a given P-point appears to have the same round shape as a Euclidean circle. Its center, however, does not always appear to lie in the Euclidean center. In fact, the farther the circle's center is from the origin the more "off-center" its center appears.

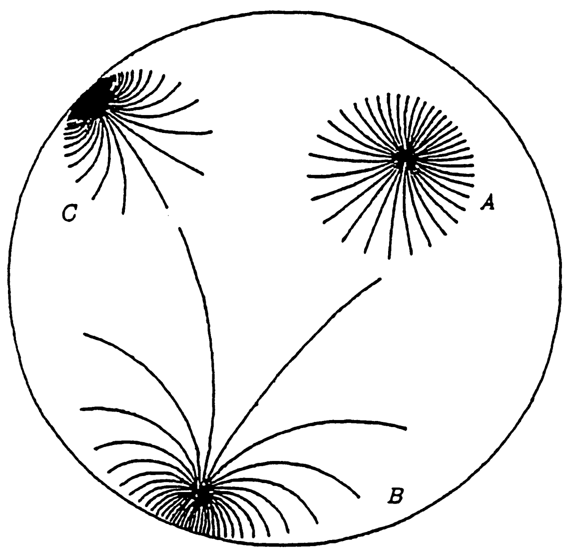


Figure 7 (three clusters of radii):

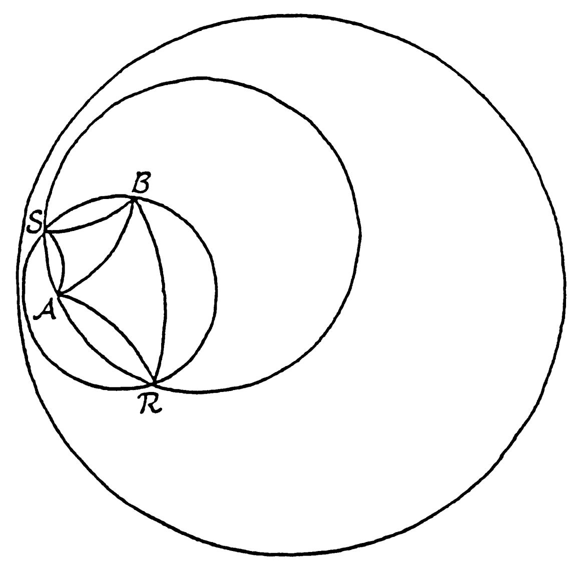


Figure 8 (Construction of Equilateral Triangles):

With this notion of a circle, we can perform many of the "ruler and compass constructions" from Euclidean Geometry. Figure 8 shows the construction of two equilateral triangles. We began the with line segment $\overline{AB}$. We then constructed a circle with center A and radius equal $d(\overline{AB})$. We also constructed a circle of the same radius with center B. Finally, we plotted points R and S at the intersections of the circles, and drew the radii $\overline{AR}, \overline{AS}, \overline{BR}, \overline{BS}$. Therefore, ΔABR and ΔABS are equilateral triangles.

Figure 9 shows 24 infinite lines which can be used to define a coordinate system in the Poincaré Model. The lines *AB* and *AH* are marked off into congruent segments of length 0.5 units. Let *AB* be called the *x-axis*, and let *AH* be called the *y-axis*. The lines drawn through each of the points along each axis are perpendicular to that axis. Consider quadrilateral □*ZCAH*. This quadrilateral has three right angles ($\angle A$, $\angle C$, $\angle H$) and one acute angle ($\angle Z$). We define the coordinates of any point in the first quadrant to be *the perpendicular distance from the point to each axis*. That is, the coordinates of *Z* are (d($\overline{ZH}$), d($\overline{ZC}$)) = (1.3, 0.9). This definition gives us a one-to-one correspondence between all of the points in the first quadrant and all ordered pairs (x,y) where x and y are positive real numbers. It is interesting to notice that defining coordinates to be *the distance along each axis to the perpendiculars which pass through the point* does not set up a one-to-one correspondence. By this definition, the coordinates of *Z* would be (d($\overline{AC}$), d($\overline{AH}$)) = (1.0, 0.5), and in fact every point would correspond to a unique set of coordinates; However, some coordinates pairs would not correspond to a point. For example, the point (1,1), would not exist because the perpendicular to the x-axis at *C* (d($\overline{AC}$) = 1) and the perpendicular to the y-axis at *I* (d($\overline{AI}$) = 1) *do not intersect*!

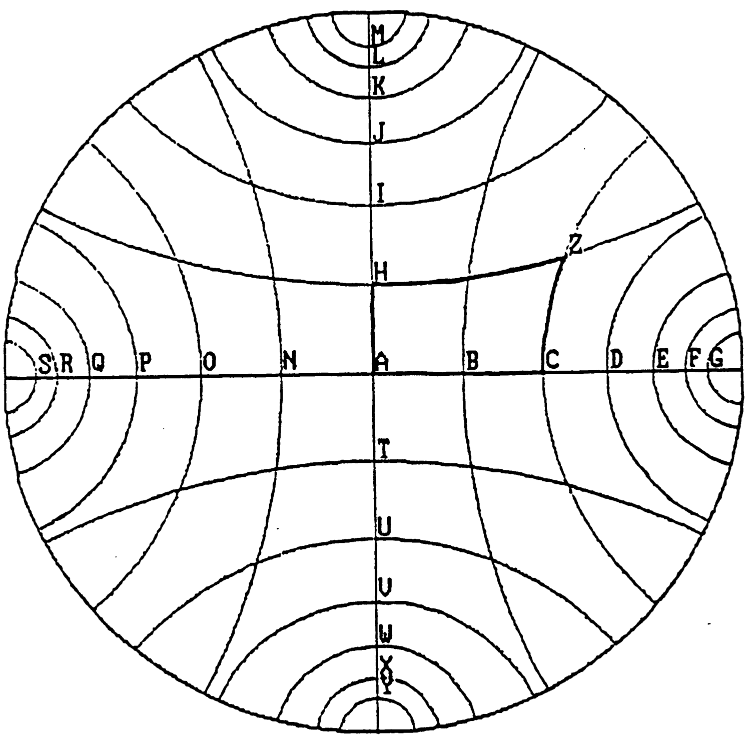


Figure 9 (Coordinate System):
The lines *AB* and *AH* are marked off into 24 congruent segments of length 0.5 units.
$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \cdots \cong \overline{XY}$
Note: the distance from *M* to the edge of the boundary circle is infinite. In fact, the distance from any point to the boundary is infinite.

Congruent Triangles:
In figure 4 we saw a tiling of congruent triangles. We will now take a closer look at congruence. The corresponding parts of ΔABC and ΔDEF in figure 10 are congruent. Therefore, the two triangles are congruent. In other words, we could move ΔABC on top of ΔDEF so that the 3 sides of each triangle perfectly coincide. Our intuition balks at this because the two triangles appear different. The catch is that the process of moving a triangle will, in some sense, distort the triangle. We can get an idea of what happens to a line as it moves through hyperbolic space by looking back to the radii clusters in figure 7. Think of one of the clusters as a strobe-light picture of a single radius that is pivoting on the circle's center. It would appear to us as if the line bends, stretches, and compresses as it moves. Actually, the line remains straight and of constant length - it is the hyperbolic plane that is curved.

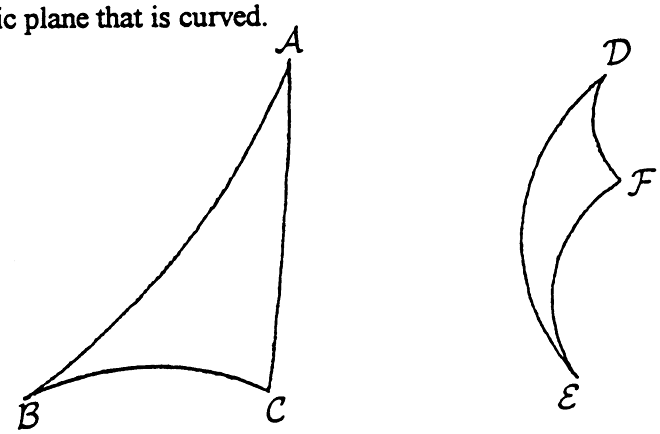


Figure 10 (Congruent Triangles):
$d(\overline{AB}) = d(\overline{DE}) = 4.0$
$d(\overline{AC}) = d(\overline{DF}) = 2.0$
$d(\overline{BC}) = d(\overline{EF}) = 3.0$

$m\angle A = m\angle D = 21°$
$m\angle B = m\angle E = 7°$
$m\angle C = m\angle F = 72°$

The SAS Postulate:
In the *Elements*, Euclid presents what he believes to be a proof for SAS[2]:

Given: ΔABC and ΔDEF, with $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$
Proof: Move ΔABC such that point A coincides with point D, and line AB coincides with DE.
The point B will coincide with E, because $\overline{AB} \cong \overline{DE}$.
Also, line AC will coincide with DF, because $\angle A \cong \angle D$.
The point C will coincide with F, because $\overline{AC} \cong \overline{DF}$.
Line BC will coincide with EF, because two lines cannot inclose a space.
Finally, $\overline{BC} \cong \overline{EF}$, because the lines and endpoints of each coincide.
Therefore, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and $\Delta ABC \cong \Delta DEF$.

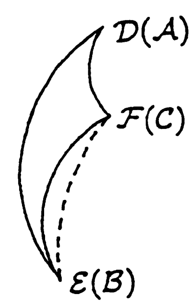


Figure 11 (Euclid's presentation of SAS):

Euclid's proof depends on the undefined term "move".

Moise[1], defines "move" (in both Euclidean and Hyperbolic geometry) to be a function that maps a set of points $P_1, P_2, P_3,\ldots$ to $P'_1, P'_2, P'_3,\ldots$, in such a way that for any two points P_n and P_m of the original set $d(\overline{P_nP_m}) = d(\overline{P'_nP'_m})$.

Adopting SAS as a postulate requires that *for any two lines* L *and* L'*, it is always possible to "move" line* L *so that it coincides with* L'. SAS is true in both Euclidean and Hyperbolic geometry.

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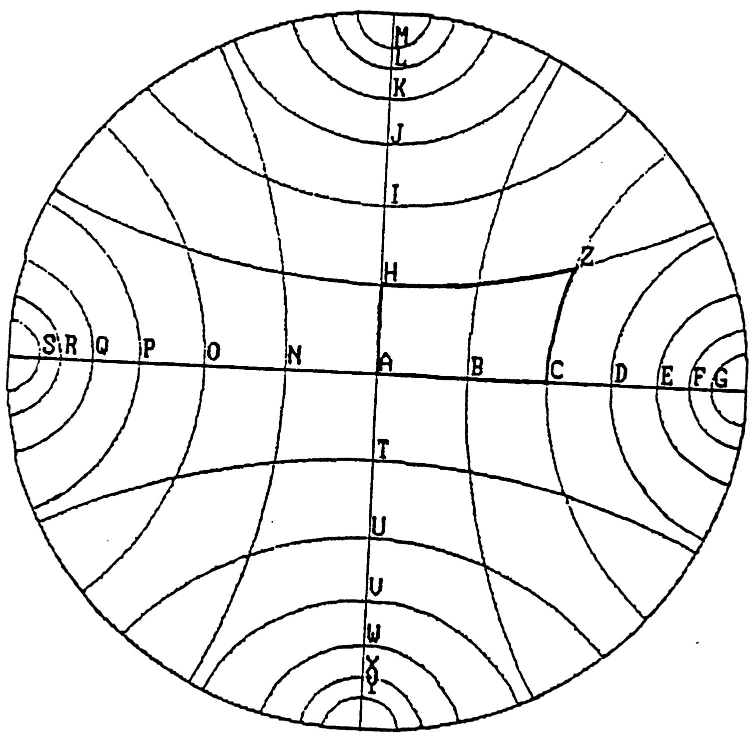

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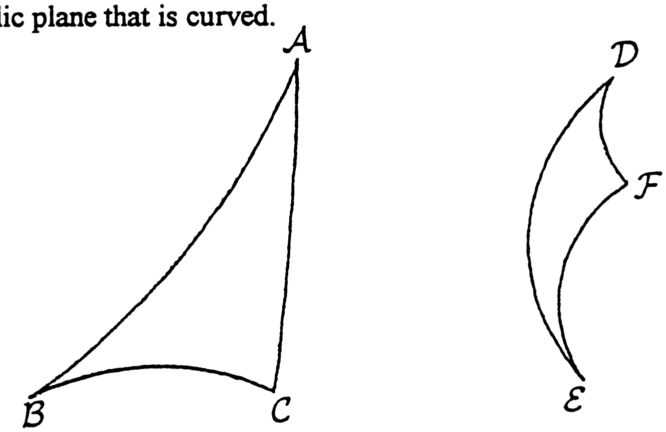

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Line BC will coincide with EF, because two lines cannot inclose a space.
Finally, $\overline{BC} \cong \overline{EF}$, because the lines and endpoints of each coincide.
Therefore, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and $\Delta ABC \cong \Delta DEF$.

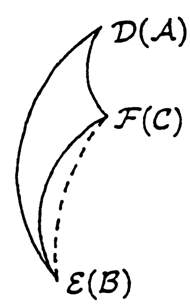

Figure 11 (Euclid's presentation of SAS):

Euclid's proof depends on the undefined term "move".

Moise[1], defines "move" (in both Euclidean and Hyperbolic geometry) to be a function that maps a set of points $P_1, P_2, P_3, \ldots$ to $P'_1, P'_2, P'_3, \ldots$, in such a way that for any two points P_n and P_m of the original set $d(\overline{P_nP_m}) = d(\overline{P'_nP'_m})$.

Adopting SAS as a postulate requires that *for any two lines* L *and* L$'$*, it is always possible to "move" line* L *so that it coincides with* L$'$. SAS is true in both Euclidean and Hyperbolic geometry.

More on Congruent Triangles:
In the Poincaré Model, it is very difficult to visualize the movement of a triangle from one place to another. Therefore, it is convenient to think of congruent triangles as a pair of triangles in a given geometry that are completely indistinguishable by any measurement possible *from within that given geometry.*

Look again at triangles ΔABC and ΔDEF of figure 10. Certainly these triangles are distinguishable by measurements that we make with our eyes. Our eyes, however, are not in the same geometry as the triangles. Figure 12 shows example measurements that a student might make to convince himself that these two triangles are congruent. The altitudes divide each triangle up into 6 non-overlapping triangles. All of the line segments and all of the angles formed in ΔABC are equal in measure to the corresponding parts formed in ΔDEF (in particular, $d(\overline{AZ}) = d(\overline{DT}) = 1.5$, $d(\overline{ZB}) = d(\overline{TE}) = 2.5$, etc.). NonEuclid allows these constructions and measurements to be made very easily.

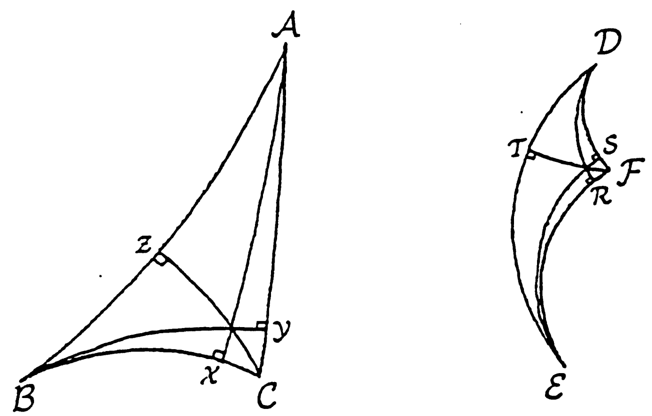

Figure 12 (Altitudes of congruent triangles):

Area:
In Euclidean Geometry, the area of a triangle is calculated by multiplying the length of any side times the corresponding height and dividing the product by two. This method does not work in Hyperbolic Geometry because the product of the base and the height is not independent of the choice of a base. For example, in ΔABC of figure 12, $d(\overline{AB}) * d(\overline{CZ}) \neq d(\overline{AC}) * d(\overline{BY}) \neq d(\overline{BC}) * d(\overline{AX})$.

In defining an area function for Hyperbolic Geometry, we should think first about what essential properties a notion of area ought to satisfy. Our intuition might first be attracted to some measurement of the Euclidean area that a figure encloses. A problem with this is that we have already seen pairs of congruent triangles which enclose different Euclidean areas. Whatever definition of area we define, it should be invariant as an object moves from one place to another.

Another property that our definition should preserve is area addition. For example, every polygonal region, both in Euclidean and Hyperbolic geometry, can be cut up into a finite number of non-overlapping triangular regions. Figure 13 shows two different ways to cut up the same polygonal region. In fact, any polygonal region can be cut up into triangular regions in infinitely many ways. Our notion of area should be such that the area of a polygonal region is equal to the sum of the areas of the triangular regions that decompose it. This implies that the sum should depend only on the region that we started with, and should be independent of the way in which we cut it up.

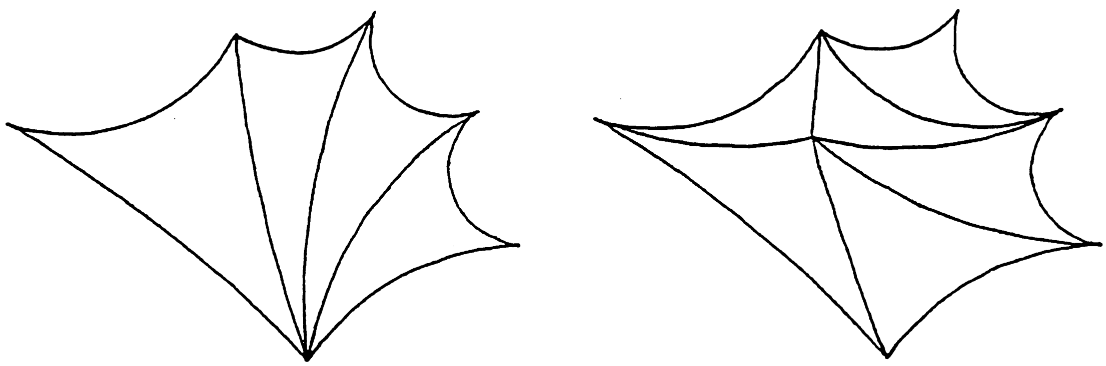

Figure 13 (Decomposing Polygons into Triangles):

One of the consequences of the postulates for Hyperbolic Geometry is that the sum of the angle measures in a triangle is always strictly less then 180°. The amount that this sum differs from 180° is used as the area function for Hyperbolic Geometry.

P-area: The P-area of a P-triangular region is 180° minus the sum of the three P-angle measures of the P-triangle. The P-area of a P-polygonal region is the sum of any set of non-overlapping P-triangular regions which completely decompose the given P-polygonal region [1].

One of the interesting consequences of this definition is that the maximum area of a triangle is bounded (the area will always be less then 180°), yet the area of an arbitrary polygon is unbounded. The area of a circle can be found by a series of inscribed and circumscribed regular polygons. Using NonEuclid a student could determine whether the P-area of a P-circle seems to be a function of pi.

Another interesting difference between Euclidean and Hyperbolic geometry is that in Hyperbolic geometry, *there does not always exist a circle passing through three given noncollinear points.*

Consider The three noncollinear points A, B, and C in figure 14. If there is a circle that passes through these three points, then its center must be equidistant from the three points. Every point that is equidistant from points A and C must lie on the perpendicular bisector of $\overline{AC}$ Likewise, every point that is equidistant from points C and B must lie on the perpendicular bisector of $\overline{BC}$. Since these two perpendicular bisectors are parallel (do not intersect), there does not exist a point that is equidistant from A, B, and C. Therefore, it is impossible to construct a P-circle which passes through points A, B, and C.

Points X, Y, and Z of figure 14 show an example of three noncollinear points through which a circle does pass. Notice that the perpendicular bisectors of $\overline{ZX}$ and $\overline{ZY}$ do intersect.

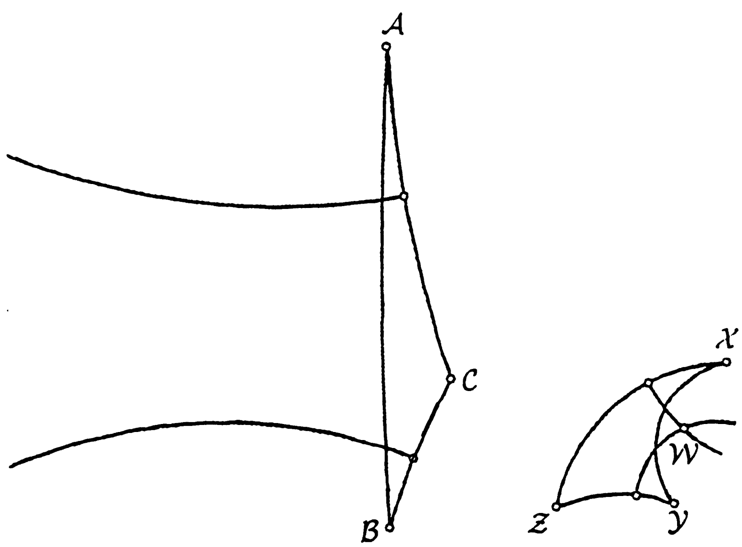

Figure 14 (Do three points determine a circle?)

The points A, B and C form an obtuse triangle. The points X, Y, and Z form an acute triangle. Is it always true that if three points form the vertices of an obtuse triangle, then a circle cannot be drawn through the points? Alternatively, it might be that acute triangles can always be circumscribed, and obtuse triangles can sometimes be circumscribed. A third possibility is that both acute and obtuse triangles can sometimes be circumscribed. Questions like this stimulate wonderful debates as some students explore the computer for counter examples and others hammer away with logic/intuition.

Appendix:

Problem: Given two points $P = (P_x, P_y)$ and $Q = (Q_x, Q_y)$ on the interior of the unit circle $\mathcal{C}$ with center at the origin of a Cartesian coordinate system, find the equation of the circle $\mathcal{C}_\perp$ which is orthogonal to $\mathcal{C}$.

Solution: Let (X_o,Y_o) be the coordinates of the center of $\mathcal{C}_\perp$, and let $r_\perp$ be the radius. Since the two points P and Q lie on $\mathcal{C}_\perp$, we have the following two equations

$$(P_x - X_o)^2 + (P_y - Y_o)^2 = r_\perp^2$$

$$(Q_x - X_o)^2 + (Q_y - Y_o)^2 = r_\perp^2.$$

Since the two circles are orthogonal, the line segment joining the two centers forms the hypotenuse of a right triangle with one leg a radius of $\mathcal{C}$ and the other a radius of $\mathcal{C}_\perp$ Thus, the Pythagorean theorem gives

$$X_o^2 + Y_o^2 = r_\perp^2 + 1^2.$$

Expanding equation (1) and using equation (3) gives

$$2P_x(X_o) + 2P_y(Y_o) = P_x^2 + P_y^2 + 1.$$

Expanding equation (2) and using equation (3) gives

$$2Q_x(X_o) + 2Q_y(Y_o) = Q_x^2 + Q_y^2 + 1.$$

Equations (4) and (5) are linear equations in X_o and Y_o since the other variables are fixed. There will be a solution when the determinant of coefficients is not zero, that is if and only if

$$4(P_yQ_x - P_xQ_y)$$

is not zero. The determinant is not zero if and only if the two points lie on a line through the origin (0,0), that is the two points lie on a diameter $\mathcal{C}$. In this case this diameter is the appropriate P-line. When the two points P and Q do not lie on a diameter of $\mathcal{C}$, equations (4) and (5) can be easily solved to give the center (X_o,Y_o) of the orthogonal circle.

Problem: Given the circle $\mathcal{C}$ with equation $X^2 + Y^2 = 1$, and an orthogonal circle $\mathcal{C}_\perp$ with equation $(X - X_o)^2 + (Y - Y_o)^2 = r_\perp^2$, find the intersection points, (A_x, A_y) and (B_x, B_y) of the two circles.

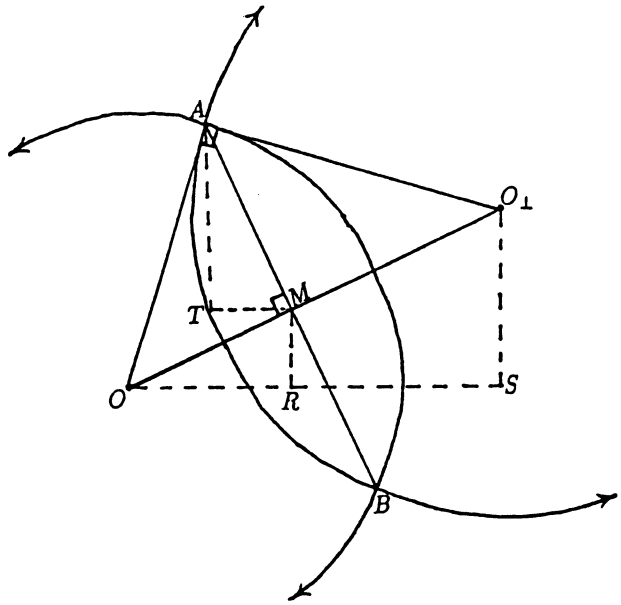

Figure 15 (Finding the intersections of $\mathcal{C}$ and $\mathcal{C}_\perp$):

Solution: In Figure 13, O is the center of unit circle $\mathcal{C}$. $O_\perp$ is the center of orthogonal circle $\mathcal{C}_\perp$. Points A and B are the intersection of $\mathcal{C}$ and $\mathcal{C}_\perp$. M is defined to be the midpoint of line segment $\overline{AB}$. This means that $\Delta OAM \cong \Delta OBM$ by SSS. Therefore, $\overline{AB} \perp \overline{OO_\perp}$

Since $\overline{AM}$ is an altitude of $\Delta AOO_\perp$, the triangles $\Delta AOO_\perp$ and ΔMOA are similar. The dotted lines shown in the figure are all parallel to either the x-axis or the y-axis of the coordinate system with origin at O. Therefore, $\Delta ORM \sim \Delta OSO_\perp \sim \Delta ATM$. From here it is easy to show that

$$A_x = \frac{X_o - r_\perp Y_o}{d^2} \qquad A_y = \frac{Y_o + r_\perp X_o}{d^2}$$

$$B_x = \frac{X_o + r_\perp Y_o}{d^2} \qquad B_x = \frac{Y_o - r_\perp X_o}{d^2}$$

where d is the length of line segment $\overline{OO_\perp}$

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