# A New Technique for Generating Quadratic Programming Test Problems

*Paul Calamai*
*Luis Vicente*
*Loaquim Judice*

**CRPC-TR92215**
**February, 1992**

Center for Research on Parallel Computation
Rice University
P.O. Box 1892
Houston, TX 77251-1892

# A New Technique for Generating Quadratic Programming Test Problems

PAUL H. CALAMAI
University of Waterloo
and
LUIS N. VICENTE
Universidade de Coimbra
and
JOAQUIM J. JÚDICE
Universidade de Coimbra

This paper describes a new technique for generating convex, strictly concave and indefinite (bilinear or not) quadratic programming problems. These problems have a number of properties that make them useful for test purposes. For example, strictly concave quadratic problems with their global maximum in the interior of the feasible domain and with an exponential number of local minima with distinct function values and indefinite and jointly constrained bilinear problems with nonextreme global minima, can be generated.
Unlike most existing methods our construction technique does not require the solution of any subproblems or systems of equations. In addition, the authors know of no other technique for generating jointly constrained bilinear programming problems.

Key Words and Phrases: test problem generation, quadratic programming, global optimization, large-scale optimization.

Abbreviated Title: Generating Quadratic Programming Test Problems.

## 1 Introduction.

The testing and development of new algorithms and the benchmarking of available software for quadratic programming and for global optimization benefit from the availability of test problems [2, 4, 5, 6]. These test problems normally come from two sources: collections of real-world problems and randomly generated problems. In the latter case the types of problems generated are often based on complexity issues. References [7, 9, 12, 16, 18, 19, 20] provide a discussion of these issues.

Several authors have proposed methods for generating quadratic programming test problems. One of the earliest of these was a technique proposed by Rosen and Suzuki [23]. Their idea was implemented by Michaels and O'Neil [15] to construct convex quadratic programs with a user specified global solution. Lenard and Minkoff [13] describe an alternative technique for randomly generating strictly convex quadratic programs in the form of linearly constrained

Authors' addresses: P. H. Calamai, University of Waterloo, Department of Systems Design Engineering, Waterloo, Ontario, Canada N2L 3G1; L. N. Vicente and J.J. Júdice, Universidade de Coimbra, Departamento de Matemática, 3000 Coimbra, Portugal.

This paper was completed when the first author was on a research sabbatical at Rice University.

Support of this work has been provided by the Instituto Nacional de Investigação Científica de Portugal (INIC) under contract 89/EXA/5 and by the Natural Sciences and Engineering Research Council of Canada operating grant 5671.

linear least-squares problems.

Other construction techniques make use of the fact that the minima of a concave quadratic function over a closed bounded polyhedron occur at the extreme points of the feasible region. In [22] Rosen presents a method for constructing concave quadratic programming problems with a global minimum at a selected vertex of a prespecified bounded convex polyhedron. Problems with the same characteristics are generated by the method proposed by Sung and Rosen [24]. Unlike Rosen's technique, which requires the solution of a single linearly constrained convex program and one linear program in $n$ variables, Sung and Rosen's technique requires the solution of $n$ linear programs in $n$ variables, where $n$ is the dimension of the constructed problem.

Kalantari and Rosen [11] and Pardalos [17] describe different methods for constructing large-scale nonconvex quadratic programs which have a global minimum at a selected nondegenerate vertex of a prespecified bounded polytope. Both methods require the solution of a linear program and a system of linear equations. One disadvantage of such test problems is that algorithms that systematically visit vertices, by finding a sequence of local solutions (or otherwise), may perform quite well on these problems but poorly on others. To address this possibility the authors in [7] propose a method for generating indefinite quadratic programs that have, as their global minimizer, an arbitrarily specified boundary point (extreme or nonextreme). An alternative approach to this possibility, proposed by Kalantari [10], involves generating box constrained concave quadratic programs with an exponential number of local minima. While the former approach is much more flexible than the latter one advantage of Kalantari's approach is that it is simple and does not require an orthogonal factorization or the solution of linear programs.

In this paper we describe a new technique for generating random quadratic programming problems that was motivated by some earlier work on generating random bilevel programming problems [3]. Our approach involves combining $m$ two-variable problems to construct a separable quadratic program in $2m$ variables. We demonstrate how convex, strictly concave and indefinite quadratic programming test problems can be constructed by simply selecting the appropriate parameters for these two-variable problems. We then show how this separability can be disguised, and randomness introduced, via a simple linear transformation of variables.

Among the features of our unified approach is the ability to generate the problems that have proven to be computationally *hard* (see, for example, [7, 8, 21]), namely;

- strictly concave quadratic programs with an exponential number of local minima (with distinct function values), with global solutions in the strict interior of the feasible domain and with a restricted number of linear variables, and

- indefinite quadratic programs (including jointly constrained bilinear problems [1]) with an exponential number of local minima and with nonextreme minima (local and/or global).

As well, the size, density and geometry of the generated problems can by controlled. Moreover, the proposed construction technique is simple and does not require the solution of linear (or convex) programs or systems of equations. In addition, the authors know of no other technique for generating jointly constrained bilinear test problems.

In section 2 we define a general quadratic programming problem in $2m$ variables and specify the set of conditions we use to replace this general problem by a disjoint set of $m$ two-variable problems. In section 3 we demonstrate how convex, strictly concave or indefinite (bilinear or not) quadratic problems can be constructed using these two-variable problems and describe the properties of the corresponding separable quadratic programming problems that result from these constructions. In section 4 we describe the transformation that is used to disguise the separability of this problem and establish an equivalence between the transformed problem and the original problem. Special considerations for large-scale test problems are then discussed in section 5. An example that illustrates the technique appears in section 6 and concluding remarks are made in section 7.

# 2 Problem Definition and Motivation.

Define quadratic programming problem $QP(Q,s,A,c)$ as:

$$\text{minimize } F(x,y) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Q^x & Q^{xy} \\ Q^{yx} & Q^y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} s^x \\ s^y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \xi$$

subject to

$$\begin{bmatrix} A^x & A^y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq c$$

where $s^x, x \in R^{n_x}$, $s^y, y \in R^{n_y}$, $Q^x \in R^{n_x \times n_x}$, $Q^y \in R^{n_y \times n_y}$, $Q^{xy} = Q^{yx^T} \in R^{n_x \times n_y}$, $A^x \in R^{\gamma \times n_x}$, $A^y \in R^{\gamma \times n_y}$, $c \in R^\gamma$ and $\xi \in R$.

Our technique for constructing test problems starts with a separable version of problem $QP(Q,s,A,c)$, exploits the separability to construct problems with favorable properties and then disguises the separability (without destroying those properties) via a nonsingular transformation of variables.

To arrive at a separable version of problem $QP(Q,s,A,c)$ we select some integer $m > 0$ and set $n_x = n_y = m$ and $\gamma = 3m$. We also make matrices $Q^x$, $Q^y$ and $Q^{xy}$ diagonal, with $Q^x = \text{diag}(q_1^x, \cdots, q_m^x)$, $Q^y = \text{diag}(q_1^y, \cdots, q_m^y)$ and $Q^{xy} = \text{diag}(q_1^{xy}, \cdots, q_m^{xy})$, and matrices $A^x$ and $A^y$ block diagonal with $m$ three-by-one blocks. With these selections problem $QP(Q,s,A,c)$ is separable in the pairs of variables $(x_l, y_l)$, $l \in M$ where $M = \{1, \ldots, m\}$, and can be rewritten as

$$\text{minimize } F(x,y) = \sum_{l \in M} \left( \frac{1}{2} q_l^x x_l^2 + \frac{1}{2} q_l^y y_l^2 + q_l^{xy} x_l y_l - s_l^x x_l - s_l^y y_l + \xi_l \right)$$

subject to

$$a_{il}^x x_l + a_{il}^y y_l \leq c_i \quad i \in \{3l-2, 3l-1, 3l\},\ l \in M,$$

where $\xi = \sum_{l \in M} \xi_l$.

In the section that follows we exploit the fact that the properties of this separable version of problem $QP(Q,s,A,c)$ are related to the way the following $m$ subproblems $SQP_l$ $(l \in M)$ are constructed

$$SQP_l: \quad \text{minimize } \tfrac{1}{2} q_l^x x_l^2 + \tfrac{1}{2} q_l^y y_l^2 + q_l^{xy} x_l y_l - s_l^x x_l - s_l^y y_l + \xi_l$$

subject to

$$a_{il}^x x_l + a_{il}^y y_l \leq c_i \quad i \in \{3l-2, 3l-1, 3l\}.$$

We also exploit the fact that if $(x_l^L, y_l^L)$ are minima of problems $SQP_l$, $l \in M$, then $(x^L, y^L) = (x_1^L \cdots x_m^L, y_1^L \cdots y_m^L)^T$ is a minima of problem $QP(Q,s,A,c)$.

# 3 Problem Construction and Properties.

In this section we define strictly concave, indefinite (bilinear or not) and convex versions of problem $QP(Q,s,A,c)$ by defining specific instances of subproblems $SQP_l$, $l \in M$.

## 3.1 Constructing Strictly Concave Problems.

Our technique for constructing a strictly concave version of problem $QP(Q,s,A,c)$ involves defining all subproblems $SQP_l$, $l \in M$, using:

$$\begin{aligned}
&q_l^x = q_l^y = q_l,\ q_l^{xy} = 0,\ s_l^x = s_l^y = s_l \text{ and} \\
&a_{l_1,l}^x = \alpha_l,\ a_{l_1,l}^y = \beta_l \text{ and } c_{l_1} = \alpha_l + \beta_l + \alpha_l\beta_l, \\
&a_{l_2,l}^x = 1,\ a_{l_2,l}^y = -(\beta_l + 1) \text{ and } c_{l_2} = 0, \\
&a_{l_3,l}^x = -(\alpha_l + 1),\ a_{l_3,l}^y = 1 \text{ and } c_{l_3} = 0,
\end{aligned}$$

where, for $\theta_l \in \{0,1\}$, we have $q_l = -\left(4^{l-L}\right)^{1-\theta_l}$, $s_l = 4^{\theta_l} q_l$ and $\xi_l = 16^{\theta_l} q_l$, and where $\{l_1, l_2, l_3\} = \{3l-2, 3l-1, 3l\}$, $\alpha_l, \beta_l \in \{3/2, 2\}$ and $\alpha_l \neq \beta_l$.

To avoid numerical difficulties in the construction of concave problems with an exponential number of local minima having distinct function values the integer parameter $L > 0$ should be chosen so that if $\theta_l = 0$ then $-4^{l-L}$ and $-2 \cdot 4^{l-L}$ are *computationally* distinguishable from 0 and $-\infty$ respectively.

With this data each subproblem $SQP_l$, $l \in M$, becomes

$$\begin{aligned}
\text{minimize } f_l(x_l, y_l) &= -\left(4^{l-L}\right)^{1-\theta_l} \left\{x_l^2/2 + y_l^2/2 - 4^{\theta_l} x_l - 4^{\theta_l} y_l + 16^{\theta_l}\right\} \\
&= -\left(4^{l-L}\right)^{1-\theta_l} \left\{(x_l - 4^{\theta_l})^2 + (y_l - 4^{\theta_l})^2\right\}/2
\end{aligned}$$

subject to

$$\begin{array}{rcrcl}
\alpha_l x_l & + & \beta_l y_l & \leq & 13/2 \\
x_l & - & (\beta_l + 1) y_l & \leq & 0 \\
-(\alpha_l + 1) x_l & + & y_l & \leq & 0.
\end{array}$$

The properties of this subproblem can be stated in terms of the value of $\theta_l$.

When $\theta_l = 0$ the point $(1,1) = \arg\max f_l(x_l, y_l)$ is in the strict interior of the feasible domain $\Omega_l$ and all vertices of $\Omega_l$ are local minima. Thus $(x_l^L, y_l^L) \in \{(1, 1+\alpha_l), (1+\beta_l, 1), (0,0)\}$ are the local minima. In addition, the restrictions on $\alpha_l$ and $\beta_l$, and the definition of $q_l$, $s_l$ and $\xi_l$, guarantee that the objective function $f_l(x_l, y_l)$ takes a unique value (in the interval $[-2 \cdot 4^{l-L}, -4^{l-L}]$) at each of these local minima. Specifically, $f_l(1, 1+\alpha_l) = -4^{l-L}\alpha_l^2/2$, $f_l(1+\beta_l, 1) = -4^{l-L}\beta_l^2/2$ and $f_l(0,0) = -4^{l-L}$. Figure 1 depicts this case (ie. $\theta_l = 0$) when $\alpha_l = 3/2$ and $\beta_l = 2$.

When $\theta_l = 1$ the point $(4,4) = \arg\max f_l(x_l, y_l)$ is outside the feasible domain $\Omega_l$. In this case, the vertex of $\Omega_l$ farthest from this point is the global minimum of problem $SQP_l$ and there are no other minima. Specifically, $(x_l^G, y_l^G) = (0,0)$ with $f_l(0,0) = -16$.

To analyze the properties of problem $QP(Q,s,A,c)$ (that result as a consequence of the properties of subproblems $SQP_l$, $l \in M$) we define the following partitions of the set $M$: $M^0 = \{l \in M : \theta_l = 0\}$ and $M^1 = \{l \in M : \theta_l = 1\}$ with cardinalities $m^0$ and $m^1$ respectively.

**Property 3.1** *Problem $QP(Q,s,A,c)$ is strictly concave and has $3^{m^0}$ local minima including an unique global minimum $(x^G, y^G)$ with function value*

$$F(x^G, y^G) = -16m^1 - 2\sum_{l \in M^0} 4^{l-L}.$$

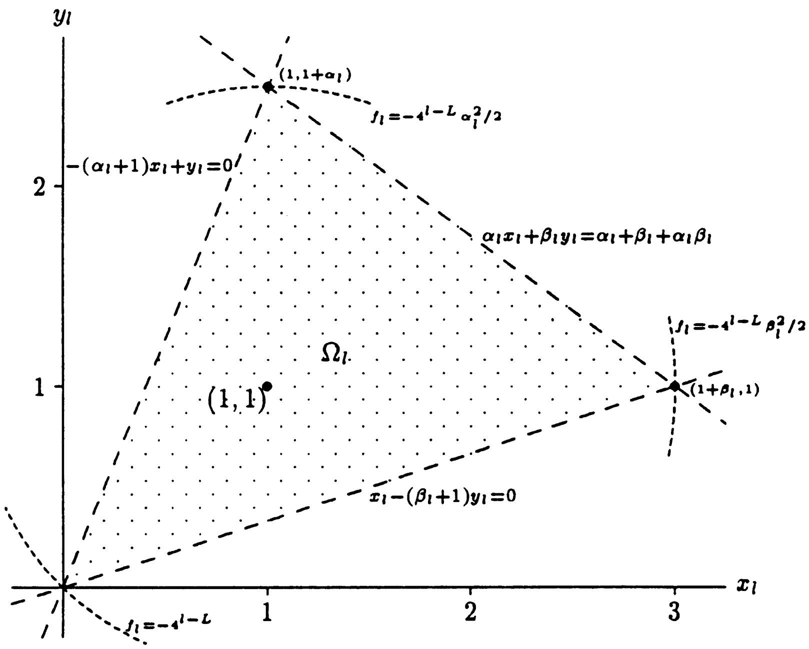


Figure 1: $SQP_l$ concave with $\alpha_l = 3/2$, $\beta_l = 2$ and $\theta_l = 0$

**Property 3.2** *All local minima of problem $QP(Q,s,A,c)$ have distinct function values.*

**Proof.** In order to simplify the proof assume, without loss of generality, that $m^1 = 0$.

If $M^0 = \{1\}$ the result follows immediately from the uniqueness of $f_1$ at the corresponding local minima $(x_1^L, y_1^L)$. Assume that the result also holds when $M^0 = \{1,\ldots,l\}$, for some $l \in \{1,\ldots,m-1\}$, and that the largest (absolute) gap in the value of the objective function $F$ among the local minima for this case is denoted by $g^{\max}$. The proof that the property holds relies on proving that $g^{\max}$ is less than $g_{l+1}^{\min}$, where $g_{l+1}^{\min}$ is defined to be the smallest (absolute) gap in the value of $f_{l+1}$ between all local minima $\{(x_{l+1}^L, y_{l+1}^L)\}$. In order to do this, define $g_i^{\max}$, $i = 1,\ldots,l$, to be the largest absolute gap in the value of $f_i$ between all local minima $\{(x_i^L, y_i^L)\}$ and note that

$$g^{\max} = \sum_{i=1}^{l} g_i^{\max} = \sum_{i=1}^{l} 3 \cdot 4^{i-L} = 4^{l+1-L} - 4^{1-L}.$$

The proof then follows since $g_{l+1}^{\min} = (5/4)4^{l+1-L} > g^{\max}$. □

**Property 3.3** *If $m^1 = 0$ then $e = \arg\max F(x,y)$ is interior to the feasible domain of problem $QP(Q,s,A,c)$, where $e$ is the ones-vector of dimension $n$. Otherwise $\arg\max F(x,y)$ is exterior to the feasible domain of problem $QP(Q,s,A,c)$.*

**Property 3.4** *The gradients of the active constraints at all minima of problem $QP(Q,s,A,c)$ are linearly independent.*

## 3.2 Constructing Jointly Constrained Bilinear Problems.

Our technique for constructing an indefinite (jointly constrained bilinear) version of problem $QP(Q,s,A,c)$ involves defining all subproblems $SQP_l$, $l \in M$,

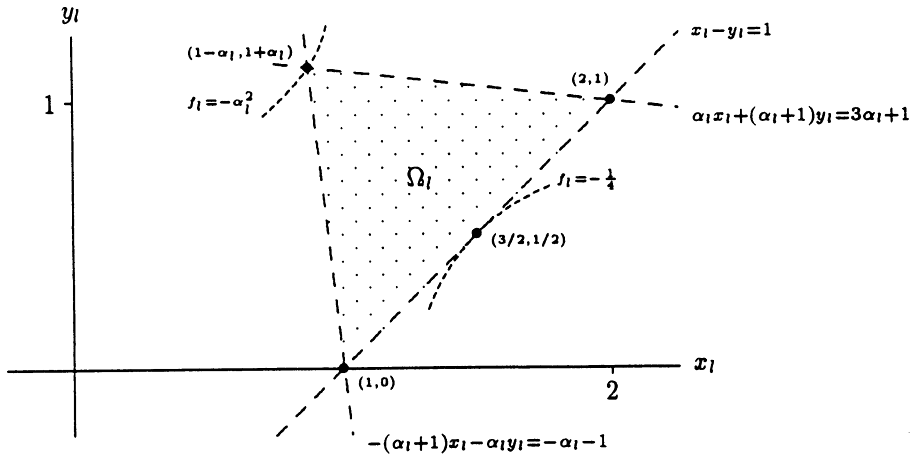


Figure 2: $SQP_l$ jointly constrained bilinear with $\alpha_l = \frac{1}{8}$

using:

$$
\begin{aligned}
&q_l^x = q_l^y = 0,\ q_l^{xy} = s_l^x = s_l^y = \xi_l = 1 \text{ and} \\
&a_{l_1,l}^x = \alpha_l,\ a_{l_1,l}^y = \alpha_l + 1 \text{ and } c_{l_1} = 3\alpha_l + 1, \\
&a_{l_2,l}^x = -(\alpha_l + 1),\ a_{l_2,l}^y = -\alpha_l \text{ and } c_{l_2} = -(\alpha_l + 1), \\
&a_{l_3,l}^x = -a_{l_3,l}^y = c_{l_3} = 1,
\end{aligned}
$$

where $\{l_1, l_2, l_3\} = \{3l-2, 3l-1, 3l\}$ and $\alpha_l > 0$. Thus $SQP_l$, for $l \in M$, becomes

$$\text{minimize } f_l(x_l, y_l) = x_l y_l - x_l - y_l + 1$$

subject to

$$
\begin{array}{rcrcl}
\alpha_l x_l & + & (\alpha_l+1)y_l & \leq & 3\alpha_l + 1 \\
-(\alpha_l+1)x_l & - & \alpha_l y_l & \leq & -(\alpha_l+1) \\
x_l & - & y_l & \leq & 1.
\end{array}
$$

As in the concave case direct observation of figure 2, which depicts problem $SQP_l$ ($l \in M$) when $\alpha_l = 1/8$, allows the following claims to be made:

- When $\alpha_l \in (0, 1/2)$ the point $(x_l^G, y_l^G) = (3/2, 1/2)$, a nonextreme point of the feasible region $\Omega_l$, is a global minima with $f_l(x_l^G, y_l^G) = -1/4$ and the point $(x_l^L, y_l^L) = (1-\alpha_l, 1+\alpha_l)$, is a local minima with $f_l(x_l^L, y_l^L) = -\alpha_l^2$.
- When $\alpha_l = 1/2$ the points $(x_l^G, y_l^G) \in \{(3/2, 1/2), (1/2, 3/2)\}$ are both global minima with $f_l(x_l^G, y_l^G) = -1/4$. The first of these is a nonextreme point of the feasible region $\Omega_l$.
- When $\alpha_l > 1/2$ the point $(x_l^G, y_l^G) = (1-\alpha_l, 1+\alpha_l)$, an extreme point of the feasible region $\Omega_l$, is a global minima with $f_l(x_l^G, y_l^G) = -\alpha_l^2$ and the point $(x_l^L, y_l^L) = (3/2, 1/2)$ is a local minima with $f_l(x_l^L, y_l^L) = -1/4$.

The following properties of problem $QP(Q, s, A, c)$, constructed using the jointly constrained bilinear subproblems $SQP_l$, $l \in M$, are expressed in terms of the partitions of the set $M$ defined by: $M^< = \{l \in M : \alpha_l < 1/2\}$, $M^= = \{l \in M : \alpha_l = 1/2\}$ and $M^> = \{l \in M : \alpha_l > 1/2\}$, with cardinalities $m^<$, $m^=$ and $m^>$ respectively.

**Property 3.5** *Problem $QP(Q,s,A,c)$ is an indefinite problem (more specific a jointly constrained bilinear problem) with $2^m$ local minima including $2^{m^=}$ global minima with function value*

$$F = -\frac{m^< + m^=}{4} - \sum_{l \in M^>} \alpha_l^2.$$

**Property 3.6** *The gradients of the active constraints at all minima of problem $QP(Q,s,A,c)$ are linearly independent.*

**Property 3.7** *If $m^< > 0$ then all global minima of the jointly constrained bilinear problem $QP(Q,s,A,c)$ are nonextreme points of the feasible domain. However, if $m^< = 0$ and $m^= > 0$ then one global minima is an extreme point.*

## 3.3 Constructing Convex Problems.

Our technique for constructing a convex version of problem $QP(Q,s,A,c)$ involves defining all subproblems $SQP_l$, $l \in M$, using:

$$\begin{aligned}
&q_l^x = 1,\ q_l^y = \rho_l,\ q_l^{xy} = 0,\ s_l^x = 3^{\theta_l},\ s_l^y = \rho_l 3^{\theta_l},\ \xi_l = (1+\rho_l)9^{\theta_l}/2 \text{ and}\\
&a_{l_1,l}^x = -3,\ a_{l_1,l}^y = -2 \text{ and } c_{l_1} = -\alpha_l,\\
&a_{l_2,l}^x = -2,\ a_{l_2,l}^y = -3 \text{ and } c_{l_2} = -\alpha_l,\\
&a_{l_3,l}^x = a_{l_3,l}^y = 1 \text{ and } c_{l_3} = 3,
\end{aligned}$$

where $\{l_1, l_2, l_3\} = \{3l-2, 3l-1, 3l\}$, $5 \le \alpha_l < 15/2$ and $\theta_l = 1 - \rho_l\omega_l$ with $\rho_l, \omega_l \in \{0,1\}$. Thus $SQP_l$, for $l \in M$, becomes

$$\begin{aligned}
\text{minimize } f_l(x_l, y_l) &= x_l^2/2 + \rho_l y_l^2/2 - 3^{\theta_l}x_l - \rho_l 3^{\theta_l} y_l + (1+\rho_l)9^{\theta_l}/2\\
&= \left\{(x_l - 3^{\theta_l})^2 + \rho_l(y_l - 3^{\theta_l})^2\right\}/2
\end{aligned}$$

subject to

$$\begin{array}{rcrcl}
-3x_l & - & 2y_l & \le & -\alpha_l\\
-2x_l & - & 3y_l & \le & -\alpha_l\\
x_l & + & y_l & \le & 3.
\end{array}$$

The properties of this subproblem depend on the values of the parameters $\theta_l$ and $\rho_l$.

When $\rho_l = 1$ the objective function $f_l(x_l, y_l)$ has quadratic and linear terms in both $x_l$ and $y_l$ and $\theta_l$ can equal either 0 or 1. Otherwise, when $\rho_l = 0$, the terms in $y_l$ vanish and $\theta_l$ must equal 0. Each of these three possibilities is discussed in greater detail below.

When $\rho_l = 1$ and $\theta_l = 0$ ($\omega_l = 1$) we have the situation depicted in figure 3 (with $\alpha_l = 6$). In such cases the point $(\alpha_l/5, \alpha_l/5)$ is the feasible point closest, in the Euclidean sense, to the point $(1,1) = \arg\min f_l(x_l, y_l)$. Thus the extreme point $(x_l^G, y_l^G) = (\alpha_l/5, \alpha_l/5)$ is the unique global minimum with $f_l(x_l^G, y_l^G) = (\alpha_l/5 - 1)^2/2$.

When $\rho_l = 1$ and $\theta_l = 1$ ($\omega_l = 0$) the point $(3/2, 3/2)$ is the feasible point closest, in the Euclidean sense, to the point $(3,3) = \arg\min f_l(x_l, y_l)$. In such cases the global minimum is the nonextreme point $(x_l^G, y_l^G) = (3/2, 3/2)$, with $f_l(x_l^G, y_l^G) = 9/4$.

Finally, when $\rho_l = 0$ (implying $\theta_l = 1$) the objective function of $SQP_l$ becomes $f_l(x_l, y_l) = (x_l - 3)^2/2$ and the unique global minimum is the extreme point $(x_l^G, y_l^G) = (9 - \alpha_l, \alpha_l - 6)$ with $f_l(x_l^G, y_l^G) = (6 - \alpha_l)^2/2$.

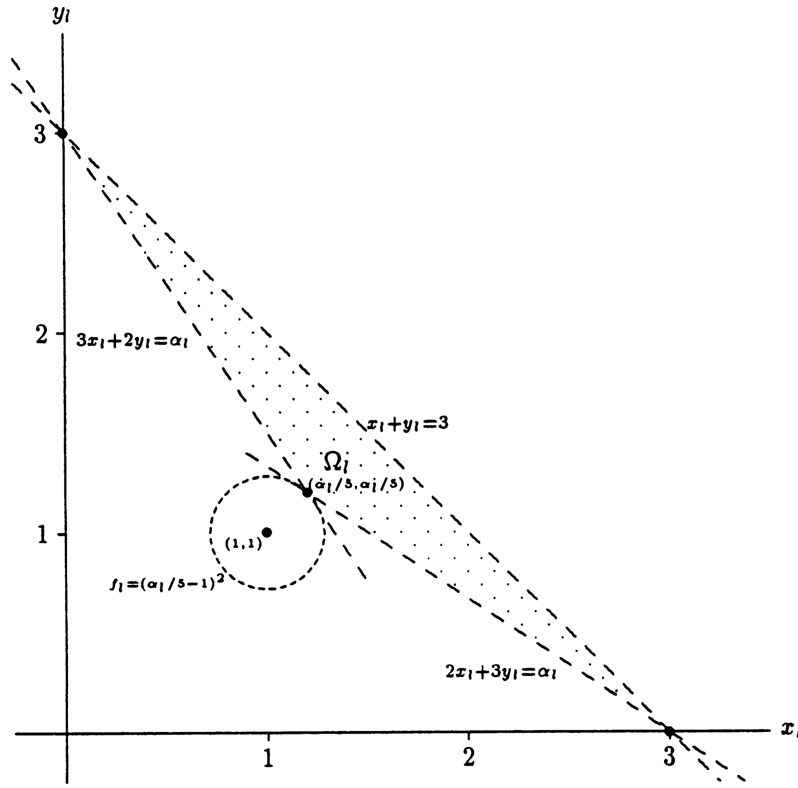


Figure 3: $SQP_l$ convex with $\alpha_l = 6$, $\theta_l = 0$ and $\rho_l = 1$

In order to state the properties of problem $QP(Q, s, A, c)$, which depend on these various cases, we define the following partition of the set $M$: $M^{0,1} = \{l \in M : \theta_l = 0 \text{ and } \rho_l = 1\}$, $M^{1,0} = \{l \in M : \theta_l = 1 \text{ and } \rho_l = 0\}$ and $M^{1,1} = \{l \in M : \theta_l = 1 \text{ and } \rho_l = 1\}$, with cardinalities $m^{0,1}$, $m^{1,0}$ and $m^{1,1}$ respectively.

Using these definitions we make the following observations regarding convex problem $QP(Q, s, A, c)$.

**Property 3.8** *Problem $QP(Q, s, A, c)$ has an unique global minimum $(x^G, y^G)$ with function value*

$$F(x^G, y^G) = 9m^{1,1}/4 + \sum_{l \in M^{1,0}} (6 - \alpha_l)^2/2 + \sum_{l \in M^{0,1}} (\alpha_l/5 - 1)^2/2.$$

*This minimum occurs at a nonextreme point of the feasible domain except when $m^{1,1} = 0$.*

**Property 3.9** *The gradients of the active constraints at $(x^G, y^G)$ are linearly independent.*

**Property 3.10** *Problem $QP(Q, s, A, c)$ is* strictly *convex when $m^{1,0} = 0$.*

## 3.4 Constructing Indefinite Problems.

In order to describe how indefinite instances of problem $QP(Q, s, A, c)$ can be constructed let set $M = \{1, ..., m\}$ be partitioned into the three sets $M_1$, $M_2$ and

$M_3$, where set $M_j$ has $m_j$ members ($j = 1, 2, 3$). Assume, for convenience only, that the sets $M_1$, $M_2$ and $M_3$ are nonempty and, without loss of generality, that $M_1 = \{1, ..., m_1\}$, $M_2 = \{m_1 + 1, ..., m_1 + m_2\}$ and $M_3 = \{m_1 + m_2 + 1, ..., m\}$. For each $l \in M_1$ define subproblem $SQP_l$ as in section 3.1. Similarly, for each $l \in M_2$ and $l \in M_3$ define subproblem $SQP_l$ as in sections 3.2 and 3.3 respectively. In this way problem $QP(Q, s, A, c)$ is composed of $m_1$ concave subproblems, $m_2$ jointly constrained bilinear subproblems and $m_3$ convex subproblems.

In order to describe the properties of problem $QP(Q, s, A, c)$ that results from this construction we use the same criteria as used in each of the proceeding subsections to: partition $M_1$ into the two sets $M_1^0$ and $M_1^1$ with cardinalities $m_1^0$ and $m_1^1$ respectively; partition $M_2$ into the three sets $M_2^<$, $M_2^=$ and $M_2^>$, with cardinalities $m_2^<$, $m_2^=$ and $m_2^>$ respectively; and to partition $M_3$ into the three sets $M_3^{0,1}$, $M_3^{1,0}$ and $M_3^{1,1}$, with cardinalities $m_3^{0,1}$, $m_3^{1,0}$ and $m_3^{1,1}$ respectively.

The following properties are exhibited by indefinite problem $QP(Q, s, A, c)$:

**Property 3.11** *Problem $QP(Q, s, A, c)$ has $3^{m_1^0} \cdot 2^{m_2^=}$ local minima including $2^{m_2^=}$ global minima with function value*

$$\begin{aligned} F &= -16m_1^1 - 2\sum_{l \in M_1^0} 4^{l-L} - \frac{m_2^< + m_2^=}{4} - \sum_{l \in M_2^>} \alpha_l^2 \\ &\quad + 9m_3^{1,1}/4 + \sum_{l \in M_3^{1,0}} (6 - \alpha_l)^2/2 + \sum_{l \in M_3^{0,1}} (\alpha_l/5 - 1)^2/2. \end{aligned}$$

*Thus if $m_2^= = 0$ problem $QP(Q, s, A, c)$ has an unique global solution.*

**Property 3.12** *If $m_2^< > 0$, or if $m_3^{1,1} > 0$, then all global minima of the indefinite problem $QP(Q, s, A, c)$ are nonextreme points of the feasible domain. However, if $m_2^= > 0$, $m_2^< = 0$ and $m_3^{1,1} = 0$, then all but one global minima occur at nonextreme points.*

**Property 3.13** *The gradients of the active constraints at all minima of problem $QP(Q, s, A, c)$ are linearly independent.*

In order to disguise the separability of problem $QP(Q, s, A, c)$ (and introduce a random component to the construction) we perform a simple transformation of variables. This transformation is described in the following section.

# 4 The Transformed Problem.

For $n = 2m$ define the order-$n$ matrix $M = DH$ where $H$ is a random Householder matrix satisfying $H = I_n - 2vv^T$, with $I_n$ the order-$n$ identity and with $v^T v = 1$ where $v \in R^n$ is sparse and random, and where $D$ is a positive definite diagonal matrix with 2-norm condition number $\kappa_2(D) = 10^\delta$ and let $W = M^{-1} = HD^{-1}$.

The following propositions characterize the relationship between problem $QP(Q, s, A, c)$ and the transformed problem, namely problem $QP(M^T QM, M^T s, AM, c)$:

**Proposition 4.1** *Problem $QP(Q, s, A, c)$ in the variables $z \in R^n$ is equivalent to problem $QP(M^T QM, M^T s, AM, c)$ in the variables $\bar{z} \in R^n$ under the nonsingular transformation $\bar{z} = Wz$.*

**Proof.** For $z = M\bar{z}$ problem $QP(Q, s, A, c)$ becomes:

$$\text{minimize } \mathcal{F}(\bar{z}) = F(M\bar{z}) = \frac{1}{2}\bar{z}^T(M^TQM)\bar{z} - s^TM\bar{z} + \xi$$

subject to

$$[AM]\bar{z} \le c$$

which is problem $QP(M^TQM, M^Ts, AM, c)$ in the variables $\bar{z} \in R^n$. □

**Proposition 4.2** *Suppose that there exists $u \in R^n : Au \le c$ such that*

$$F(u) - F(z) < 0$$

*for all $\{z \in R^n : z \ne u,\ Az \le c,\ ||u - z||_2 \le \epsilon\}$ for some $\epsilon > 0$ (ie. $u$ is a strong local minimum of problem $QP(Q, s, A, c)$). Then $Wu$ is a strong local minimum of problem $QP(M^TQM, M^Ts, AM, c)$.*

**Proof.** We have $[AM]Wu = Au \le c$ which establishes the feasibility of the point $Wu$ for problem $QP(M^TQM, M^Ts, AM, c)$. Now for the points $\{\bar{z} \in R^n : \bar{z} \ne Wu,\ [AM]\bar{z} \le c,\ ||Wu - \bar{z}||_2 \le \epsilon/||M||_2\}$ we have

$$\begin{aligned} ||u - z||_2 &= ||M(Wu - \bar{z})||_2 \text{ where } M\bar{z} = z \\ &\le ||M||_2||Wu - \bar{z}||_2 \\ &\le \epsilon, \text{ with } z \ne u, \end{aligned}$$

which by the assumption of the proposition implies that $F(u) - F(z) < 0$. Thus, by proposition 4.1, we have $\mathcal{F}(Wu) - \mathcal{F}(\bar{z}) < 0$, where $\bar{z} = Wz$. □

Using a similar argument we can establish the following proposition:

**Proposition 4.3** *If $\bar{u} \in R^n$ is a strong local minimum of problem $QP(M^TQM, M^Ts, AM, c)$ then $M\bar{u}$ is a strong local minimum of problem $QP(Q, s, A, c)$.*

Thus $\bar{z}^G = Wz^G$ is a global minima of problem $QP(M^TQM, M^Ts, AM, c)$ provided $z^G \in R^n$ is a global minima of problem $QP(Q, s, A, c)$ and this one-to-one correspondence holds for all minima.

**Remark 4.1** *Two parameters of the transformation have a direct influence on the structure of the problems generated. The sparsity of the vector $v$ controls the sparsity of $M$ (and consequently the sparsity of the data that defines problem $QP(M^TQM, M^Ts, AM, c)$) and the spectrum of $D$ influences the spectrum of $M$ (and consequently affects the spectrum of $M^TQM$ and the geometry of problem $QP(M^TQM, M^Ts, AM, c)$).*

The next two propositions help clarify these relationships.

**Proposition 4.4** *If $\eta \in [1, n]$ equals the number of nonzeros in the Householder generator $v$ then the transformation matrix $M$ has, at most, $\eta_M = \eta^2 + (n - \eta)$ nonzeros. Similarly, the matrix $AM$ has, at most, $3\eta_M$ nonzeros and the matrix $M^TQM$ has, at most, $\tau\eta_M$ nonzeros, where $\tau = 1$ (alternatively $\tau = 2$) when $m_2 = 0$ ($m_2 > 0$).*

**Proposition 4.5** *The following relationship exists between the 2-norm condition number of matrix $M^TQM$ and the magnitudes of $d_l$, $q_l^x$ and $q_l^y$, $l \in M$.*

$$\begin{aligned}\kappa_2(M^TQM) &= \kappa_2(HDQDH)\\ &= \kappa_2(DQD)\\ &= \kappa_2(B)\end{aligned}$$

*where $B = diag(B_1 \cdots B_m)$, a symmetric permutation of $DQD$, is a block diagonal matrix with $B_l = diag(d_l^2 q_l^x, d_{m+l}^2 q_l^y)$, when $l \in M_1 \cup M_3$, and $B_l = skew(d_l d_{m+l}, d_l d_{m+l})$ (where the operator $skew(\cdot)$ infers the skew diagonal), when $l \in M_2$.*

*Thus $\kappa_2(M^TQM) = B_{\max}/B_{\min}$, where*

$$B_{\max} = \max\left\{\{d_l^2|q_l^x|, d_{m+l}^2|q_l^y| : l \in M_1 \cup M_3\} \cup \{d_l d_{m+l} : l \in M_2\}\right\},$$

$$B_{\min} = \min\left\{\{d_l^2|q_l^x|, d_{m+l}^2|q_l^y| : l \in M_1 \cup M_3\} \cup \{d_l d_{m+l} : l \in M_2\}\right\},$$

*and where (by convention) $\kappa_2(M^TQM) = \infty$ when $B_{\min} = 0$ (ie. if, and only if, $m_3^{0,1} = 0$).*

*Furthermore, if $\lambda$ is an eigenvalue of the matrix $DQD$ with corresponding eigenvector $\epsilon_\lambda$ then $\lambda$ is an eigenvalue of $M^TQM$ with corresponding eigenvector $H\epsilon_\lambda$.*

**Remark 4.2** *When problem $QP(Q,s,A,c)$ is bilinear the transformed problem $QP(M^TQM, M^Ts, AM, c)$ is indefinite but not bilinear (since, unlike $M$, the matrix $M^TQM$ is not skew block diagonal). In order to preserve jointly constrained bilinear problems the transformation has to be modified so that the transformation matrix $M$ is block diagonal with two order-$m$ blocks. This is easily accomplished by setting $M = diag(M^x, M^y)$ where $M^x$ and $M^y$ are two order-$m$ transformation matrices constructed using the process previously described. If the Householder generators for $M^x$ and $M^y$ have the same zero/nonzero pattern with $\eta \in [1,m]$ nonzeros each then $M$ and $M^TQM$ will have, at most, $2\eta_M$ nonzeros, where $\eta_M = \eta^2 + (m-\eta)$, and $AM$ will have, at most, $3\eta_M$ nonzeros.*

# 5 Special Considerations for Large-scale Test Problems

Problem $QP(M^TQM, M^Ts, AM, c)$ has $n$ $(= n_x + n_y = 2m)$ variables, $\gamma$ $(= 3m)$ constraints and no more than $\tau\eta_M$ quadratic terms in $\mathcal{F}$, where $\eta_M$ is defined as in section 4 and $\tau = 1$ (alternatively $\tau = 2$) when $m_2 = 0$ $(m_2 > 0)$.

In order to produce large-scale test problems suitable for some solution techniques it may be desirable to reduce either $\gamma/n$ or $\tau\eta/n$ (or both).

One way of reducing the ratio $\gamma/n$ is to reduce the number of constraints (ie. reduce $\gamma$). This can be accomplished by eliminating any (or all) *noncrucial* constraints in each of the subproblems $SQP_l$, $l \in M_1 \cup M_3$, used in constructing problem $QP(M^TQM, M^Ts, AM, c)$. Here *noncrucial* should be interpreted to mean those constraints that are nonbinding in all global and local solutions of the corresponding subproblems.

Another possibility, which also reduces the second ratio $\tau\eta_M/n$, is to introduce additional two-variable *linear* subproblems (into the construction of problem $QP(Q,s,A,c)$), say $SQP_l$ in $(x_l, y_l)$ for $l > m$, having linear objective functions and fewer than 3 linear constraints in each of the two-variable pairs $(x_l, y_l)$ respectively.

The one drawback to both of these approaches is that the feasible region of problem $QP(M^TQM, M^Ts, AM, c)$ will no longer be closed.

Another aspect of the technique that may be significant when constructing large-scale problems has to do with the integer parameter $L$ associated with concave subproblems (see section 3.1). This parameter limits the size of $m^0 \equiv m_1^0$ which, in turn, controls the number of local minima. In this sense, increasing $L$ (and $m^0$) allows for greater problem complexity. Unfortunately, there is an upper bound on $L$ that results from the restriction given in section 3.1. This restriction can be relaxed in several ways while maintaining the integrity of the proposed approach. One way, which is in keeping with the proof of uniqueness given for property 3.1, involves redefining subproblem $f_l$, $l \in M^0$, to be:

$$f_l(x_l, y_l) = -\left(3^{l-L}\right)^{1-\theta_l}\left\{(x_l - 3^{\theta_l})^2 + (y_l - 3^{\theta_l})^2\right\}/2,$$

setting $\alpha_l, \beta_l \in \{\sqrt{3}, 2\}$, $\alpha \neq \beta$ and choosing $L$ so that $-3^{l-L}$ and $-2 \cdot 3^{l-L}$ are *computationally* distinguishable from 0 and $-\infty$. (The disadvantage of this approach is that it involves working with $\sqrt{3}$.) Another approach, with its foundation in number theory, involves removing $L$ from the formulation and redefining the parameters $(q_l, s_l, \xi_l)$, $l \in M^0$, so that the relationship between $g^{\max}$ and $g_{l+1}^{\min}$ (as defined in the proof of property 3.1) is maintained.

# 6 A Simple Example.

The following example demonstrates how this method can be used to generate quadratic programming problems.

Suppose the following values were chosen:

$$m = 3,\ n = 2m = 6,\ m_1 = 2,\ m_2 = 1,\ \text{and}\ m_3 = 0.$$

This would yield a problem with 6 (= $2m$) variables and 9 (= $3m$) constraints. The untransformed problem would be composed of $m_1 = 2$ concave subproblems (one in variables $x_1$ and $y_1$ and the other in $x_2$ and $y_2$) and $m_2 = 1$ indefinite (jointly constrained bilinear) subproblem (in variables $x_3$ and $y_3$). Consequently the overall (separable) untransformed problem would be an indefinite quadratic programming problem.

Suppose that, in addition to these values, the following data was used for the remaining parameters for the two concave subproblems:

$$\alpha_1 = 1.5,\ \beta_1 = 2,\ \alpha_2 = 2,\ \beta_2 = 1.5,\ \theta_1 = \theta_2 = 0\ \text{and}\ L = 1,$$

which correspond to $m_1^0 = 2$, $m_1^1 = 0$, $(q_1, s_1, \xi_1) = (-1, -1, -1)$ and $(q_2, s_2, \xi_2) = (-4, -4, -4)$, and that the remaining parameter (for the indefinite subproblem) is $\alpha_3 = 0.5$, which corresponds to $m_2^< = m_2^> = 0$ and $m_2^= = 1$.

The untransformed indefinite quadratic programming problem, problem $QP(Q, s, A, c)$, that would result from these specifications would be:

$$\begin{aligned}\text{minimize } F(x,y) \quad = \quad & -\frac{1}{2}(x_1-1)^2 - \frac{1}{2}(y_1-1)^2 - 2(x_2-1)^2 - 2(y_2-1)^2 \\ & + x_3y_3 - x_3 - y_3 + 1\end{aligned}$$

subject to

$$
\begin{array}{rrrrrrcr}
1.5x_1 & & & +\,2y_1 & & & \leq & 6.5 \\
x_1 & & & -\,3y_1 & & & \leq & 0 \\
-2.5x_1 & & & +\;\;y_1 & & & \leq & 0 \\
& 2x_2 & & & +\,1.5y_2 & & \leq & 6.5 \\
& x_2 & & & -\,2.5y_2 & & \leq & 0 \\
& -3x_2 & & & +\;\;\;y_2 & & \leq & 0 \\
& & 0.5x_3 & & & +\,1.5y_3 & \leq & 2.5 \\
& & -1.5x_3 & & & -\,0.5y_3 & \leq & -1.5 \\
& & x_3 & & & -\;\;\;y_3 & \leq & 1.0.
\end{array}
$$

Since $m_1 = 2$ and $m_2 = 1$ this problem has $(3^{m_1} \cdot 2^{m_2} =)$ 18 local minima including $(2^{m_2} =)$ 2 global minima; namely $(x^G, y^G) = [\ 3 \quad 1 \quad 1.5 \quad 1 \quad 3 \quad 0.5\ ]$ and $(x^G, y^G) = [\ 3 \quad 1 \quad 0.5 \quad 1 \quad 3 \quad 1.5\ ]$ with value $F(x^G, y^G) = -(1/2)4 - (2)4 - (1/4) = -41/4$.

If the following data was used in the transformation:

$$v^T = [\ 0.5 \quad 0 \quad 0.7 \quad 0.1 \quad 0.5 \quad 0\ ]$$

$$D = \text{diag}(50, 10, 10, 50, 10, 10).$$

then the resulting transformed indefinite quadratic programming problem, problem $QP(M^TQM, M^Ts, AM, c)$, would be:

$$
\min \mathcal{F}(\bar{x}, \bar{y}) = \frac{1}{2}
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix}^T
\begin{bmatrix}
-750 & 0 & 700 & 350 & 700 & -70 \\
0 & -400 & 0 & 0 & 0 & 0 \\
700 & 0 & -1470 & 140 & -770 & 2 \\
350 & 0 & 140 & -2430 & 140 & -14 \\
700 & 0 & -770 & 140 & -750 & -70 \\
-70 & 0 & 2 & -14 & -70 & 0
\end{bmatrix}
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix}
$$

$$
- \begin{bmatrix} -7.0 \\ -40.0 \\ 70.2 \\ -41.4 \\ 3.0 \\ 10.0 \end{bmatrix}^T
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} - 4
$$

subject to

$$
\begin{array}{rrrrrrcr}
27.5\bar{x}_1 & & -\,66.5\bar{x}_3 & +\;\;90.5\bar{y}_1 & -\,47.5\bar{y}_2 & & \leq & 6.5 \\
40.0\bar{x}_1 & & -\,14.0\bar{x}_3 & -\,152.0\bar{y}_1 & -\,10.0\bar{y}_2 & & \leq & 0.0 \\
-67.5\bar{x}_1 & & +\,80.5\bar{x}_3 & +\;\;61.5\bar{y}_1 & +\,57.5\bar{y}_2 & & \leq & 0.0 \\
-7.5\bar{x}_1 & 20\bar{x}_2 & -\,10.5\bar{x}_3 & -\;\;1.5\bar{y}_1 & +\;\;7.5\bar{y}_2 & & \leq & 6.5 \\
12.5\bar{x}_1 & 10\bar{x}_2 & +\,17.5\bar{x}_3 & +\;\;2.5\bar{y}_1 & -\,12.5\bar{y}_2 & & \leq & 0.0 \\
-5.0\bar{x}_1 & -\,30\bar{x}_2 & -\;\;7.0\bar{x}_3 & -\;\;1.0\bar{y}_1 & +\;\;5.0\bar{y}_2 & & \leq & 0.0 \\
-3.5\bar{x}_1 & & +\;\;0.1\bar{x}_3 & -\;\;0.7\bar{y}_1 & -\;\;3.5\bar{y}_2 & +\,15\bar{y}_3 & \leq & 2.5 \\
10.5\bar{x}_1 & & -\;\;0.3\bar{x}_3 & +\;\;2.1\bar{y}_1 & +\,10.5\bar{y}_2 & -\;\;5\bar{y}_3 & \leq & -1.5 \\
-7.0\bar{x}_1 & & +\;\;0.2\bar{x}_3 & -\;\;1.4\bar{y}_1 & -\;\;7.0\bar{y}_2 & -\,10\bar{y}_3 & \leq & 1.0.
\end{array}
$$

The two (transformed) global minima for problem $QP(M^TQM, M^Ts, AM, c)$ are: $(\bar{x}^G, \bar{y}^G) = [\ -0.277 \quad 0.1 \quad -0.2518 \quad -0.0374 \quad 0.013 \quad 0.05\ ]$ and $(\bar{x}^G, \bar{y}^G) = [\ -0.157 \quad 0.1 \quad -0.2538 \quad -0.0234 \quad 0.083 \quad 0.15\ ]$ with value $\mathcal{F}(\bar{x}^G, \bar{y}^G) = -41/4$.

## 7 Concluding Remarks.

Care must be exercised in the testing and benchmarking of algorithms and in the interpretation and dissemination of the corresponding results [4, 5, 14]. Part of the challenge is in selecting a set of problems on which the experiments will be conducted. This paper describes a computationally efficient and unified approach for generating a broad range of quadratic programming test problems with a number of user adjustable features including; the problem size and density, the number and type of minima and the geometry and curvature of the objective. A Fortran 77 code that implements this approach can be obtained by sending an e-mail request to

*phcalamai@dial.waterloo.edu*

## References


[1] F. Al-Khayyal, *Jointly constrained bilinear programs and related problems: an overview*, Computers and Mathematics with Applications, 19 (1990), pp. 53–62.

[2] R. Bartels and N. Mahdavi-Amiri, *On generating test problems for nonlinear programming algorithms*, SIAM Journal on Scientific and Statistical Computing, 7 (1986), pp. 769–798.

[3] P. Calamai and L. Vicente, *Generating quadratic bilevel programming problems*, Department of Systems Design 189-O-051191, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1, 1991.

[4] H. Crowder, R. Dembo, and J. Mulvey, *Reporting computational experiments in mathematical programming*, Mathematical Programming, 15 (1978), pp. 316–329.

[5] ——, *On reporting computational experiments with mathematical software*, ACM Transactions on Mathematical Software, 5 (1979), pp. 193–203.

[6] C. Floudas and P. Pardalos, *A collection of test problems for constrained global optimization*, vol. 455 of Lecture Notes in Computer Science, Springer-Verlag, Berlin - New York, 1990.

[7] W. Hager, P. Pardalos, I. Roussos, and H. Sahinoglou, *Active constraints, indefinite quadratic test problems, and complexity*, Journal of Optimization Theory and Applications, 68 (1991), pp. 499–511.

[8] J. Júdice and A. Faustino, *A computational analysis of LCP methods for bilinear and concave quadratic programming.*, Computers and Operations Research, 18 (1991), pp. 645–654.

[9] B. Kalantari, *Construction of difficult linearly constrained concave minimization problems*, Operations Research, 33 (1985), pp. 222–227.

[10] ——, *Quadratic functions with exponential number of local maxima*, Operations Research Letters, 5 (1986), pp. 47–49.

[11] B. Kalantari and J. Rosen, *Construction of large-scale global minimum concave quadratic test problems*, Journal of Optimization Theory and Applications, 48 (1986), pp. 303–313.

[12] M. Kozlov, S. Tarasov, and L. Hačijan, *Polynomial solvability of convex quadratic programming*, Soviet Mathematics Doklady, 20 (1979), pp. 1108–1111.

[13] M. Lenard and M. Minkoff, *Randomly generated test problems for positive definite quadratic programming*, ACM Transactions on Mathematical Software, 10 (1984), pp. 86–96.

[14] F. Lootsma, *Comparative performance evaluation, experimental design, and generation of test problems in nonlinear optimization*, in Computational mathematical programming, NATO Adv. Sci. Inst. Ser., Springer-Verlag, Berlin - New York, 1985, pp. 249–260.

[15] W. Michaels and R. O'Neill, *A mathematical program generator MPGENR*, ACM Transactions on Mathematical Software, 6 (1980), pp. 31–44.

[16] K. Murty and S. Kabadi, *Some NP-complete problems in quadratic and linear programming*, Mathematical Programming, 39 (1987), pp. 117–129.

[17] P. Pardalos, *Generation of large-scale quadratic programs for use as global optimization test problems*, ACM Transactions on Mathematical Software, 13 (1987), pp. 133–137.

[18] P. Pardalos and J. Rosen, *Constrained global optimization: algorithms and applications*, vol. 268 of Lecture Notes in Computer Science, Springer-Verlag, Berlin - New York, 1987.

[19] P. Pardalos and G. Schnitger, *Checking local optimality in constrained quadratic programming is NP-hard*, Operations Research Letters, 7 (1988), pp. 33–35.

[20] P. Pardalos and S. Vavasis, *Quadratic programming with one negative eigenvalue is NP-hard*, Journal of Global Optimization, 1 (1991), pp. 15–22.

[21] A. Phillips and J. Rosen, *A parallel algorithm for constrained concave quadratic global minimization*, Mathematical Programming, 42 (1988), pp. 421–448.

[22] J. Rosen, *Global minimization of a linearly constrained concave function by partition of feasible domain*, Mathematics of Operations Research, 8 (1983), pp. 215–230.

[23] J. Rosen and S. Suzuki, *Construction of nonlinear programming test problems*, Communications of the ACM, 8 (1965), p. 113.

[24] Y. Sung and J. Rosen, *Global minimum test problem construction*, Mathematical Programming, 24 (1982), pp. 353–355.